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Research Statement
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Overview. My research efforts are guided by the search for regularities in collections of moduli spaces. Moduli spaces are geometric spaces that parametrize selections of mathematical objects with common properties. Each point on a moduli space corresponds to a distinct algebraic equation—examining the whole yields insight into the particular, and vice-versa. Moduli spaces lie at the heart of geometry, from projective spaces and Grassmannians, to more esoteric examples, like Hilbert schemes, Quot schemes, moduli stacks, and more.

My main research contributions broaden our knowledge of the geometry and geography of Hilbert schemes and apply jet schemes in log geometry. Hilbert schemes parametrize subschemes in projective space with specified invariants. They appear throughout geometry and connect with algebra, combinatorics, representation theory, and mathematical physics. My Ph.D. thesis [18] and its extension [19] describe previously unknown geometric regularities pervading the set of all Hilbert schemes. Jet schemes parametrize jets, which we use to classify certain singularities in [9].

My long-term goals are to develop and understand the connections between the geometry of moduli spaces and related topics, such as commutative algebra and combinatorics. I plan to continue refining my categorization of Hilbert schemes, to generalize it to other moduli spaces, and to explore related math. To accomplish this, I will develop new theoretical and computational methods for studying moduli spaces. I also plan to concurrently build computer packages to support the explorations of other mathematicians and scientists.

Geography of Hilbert schemes. Defined by Grothendieck [7], Hilbert schemes are fundamental parameter spaces. Some specific examples of Hilbert schemes are well-understood, yet surprisingly little is known about their geometry in general. We know that Hilbert schemes are always connected [8] and contain a smooth point [16], but beyond this general results are scarce. By "Murphy's Law" [21] there exist Hilbert schemes with arbitrarily complicated singularities. So there is a gap in our knowledge between tractable examples and known pathologies. Thus arise natural questions:

Question 0.1. *What are the geometric properties of typical Hilbert schemes?*

Question 0.2. *Are there transitional regions between "good" and "bad" Hilbert schemes?*

Given Murphy's Law, it seems that basic properties of individual Hilbert schemes—smoothness, irreducibility, numbers of components, etc.—will be typically difficult to understand. Seeking new approaches to address these questions is a core motivation for my research.

As shown by Macaulay [12], Hilbert polynomials of homogeneous ideals are combinatorially classified by expressions of the form $p(t) = \sum_{j=1}^{r} \binom{t+b_j-j+1}{b_j}$, where $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_r \geqslant 0$ are integers. In [19] I prove the following algebraic result:

Theorem 0.3. *The lexicographic ideal is the unique saturated Borel ideal of codimension* c *with Hilbert polynomial* p *if and only if (i)* $b_r > 0$*, (ii)* $c \geqslant 2$ *and* $r \leqslant 2$*, or (iii)* $c = 1$ *and either* $b_1 = b_r$ *or* $r - s \leqslant 2$*, where* $b_1 = b_2 = \cdots = b_s > b_{s+1} \geqslant \cdots \geqslant b_r$.

This extends the main result in my Ph.D. thesis [18], written under the supervision of Greg Smith (Queen's University), and generalizes a result of Gotzmann [5]. Because any homogeneous ideal with Hilbert polynomial p defines a point on a Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^n)$, this theorem leads to the following answer to Question 0.1, providing a counterpoint to Murphy's Law:

Theorem 0.4. *A random Hilbert scheme is smooth and irreducible with probability* > 0.5.

These results stem from my experimentations on the local geometry of specific Hilbert schemes, inspired by the concrete approaches of [15, 20]. Ideally, we would have a suite of methods that, given any Hilbert scheme, return all its relevant geometric data. This goal is unreasonably optimistic however, so my approach in [18, 19] is as follows. Sorting Hilbert schemes according to Macaulay's expressions determines an infinite family of binary trees, with Hilbert schemes as nodes. The edges connecting a parent node to its children are defined by two operations $\Psi\colon p \mapsto p+1$ and $\Phi\colon p \mapsto \sum_{j=1}^{r} \binom{t+b_j-j+2}{b_j+1}$. I relate the tree structure to the generation of Borel-fixed points, which serves as a proxy for geometric complexity. The trees are further endowed with natural probability distributions. This allows me to study the density of well-behaved Hilbert schemes in the trees.

Given $p(t)$ as above and $n > \deg p$, Theorem 0.3 has the following concise corollary, which provides a partial answer to Question 0.2:

Corollary 0.5. *If* $b_r > 0$ *holds, then the Hilbert scheme* $\mathrm{Hilb}^p(\mathbb{P}^n)$ *is smooth and irreducible.*

More remains to be studied. Does terminating a tree-path with some minimal number of Ψ-edges ensure maximal radius or multiple components in the corresponding Hilbert scheme? This generalizes the well-known open question on the minimal number n of points such that $\mathrm{Hilb}^n(\mathbb{C}^3)$ is reducible. Is the complexity of singularities restricted by bounding the number of Ψ-edges? Can we generalize the geography to multigraded/toric/tropical Hilbert schemes, or other moduli spaces? More information about the geometry of Hilbert schemes remains to be discovered in these trees and I envision these questions leading to new research.

Log-jet schemes and singularities. In [9] Kalle Karu (University of British Columbia) and I initiate the study of singularities of pairs via jet schemes in the category of log schemes. Jets are higher-degree tangent vectors—infinitesimal curves approximating other spaces—that are parametrized by jet schemes. Jet schemes have fruitful applications to singularity theory; for example, Mustaţă proved that all jet schemes over a local complete intersection variety are irreducible if and only if the variety has canonical singularities [13]. Singularities of pairs and the log-canonical threshold also have geometric manifestations in jet schemes [14]. Singularities of pairs are an important generalization of singularities of varieties and of divisors in smooth varieties. However, Kollár identifies in [11] the need for a suitable category for their study. Log geometry is developed in [10] for different purposes, but provides a category for related pairings. This leads to the following question:

Question 0.6. *Is the log category a suitable framework for studying singularities of pairs?*

We give an affirmative answer, by defining local complete intersection varieties and canonical singularities in the log category, and relating canonical log varieties to canonical and log-canonical pairs. Our main result is a log geometry analogue of Mustaţă's theorem [13]:

Theorem 0.7. *Let* $\mathbf{X}$ *be a normal local complete intersection log variety defined over an algebraically closed field of characteristic* 0. *Then* $\mathbf{X}$ *has canonical singularities if and only if the log jet scheme* $J_m(\mathbf{X})$ *is irreducible for all* $m > 0$.

This demonstrates that the category of log schemes is a viable framework for studying singularities in higher dimensional geometry. Further progress is made in this direction in [4].

Macaulay2 development. During my doctorate I became an experienced user of the computer algebra program *Macaulay2*. *Macaulay2* is a free and open-source program supporting research in algebraic geometry and commutative algebra [6]. This program is now a key experimental tool in my research. I have written various methods and packages to aid my research, allowing me to generate and reproduce well-known results and to discover new mathematical phenomena. *Macaulay2* relies on volunteers to contribute to its development. In addition to my mathematical research, I am learning about package development for the *Macaulay2* environment. I believe that this will not only enhance my research, but will also benefit the broader mathematical and scientific communities, as *Macaulay2* needs younger users to evolve into developers in order to keep up with the cutting edge of computing and mathematical research.

Superpotentials and Hilbert schemes. Superpotentials are special parameters used to solve fundamental equations in high energy physics. Computing superpotentials is an important step in understanding quiver gauge theories on Calabi-Yau threefolds [1, 17]. Not only are superpotentials important in theoretical physics, but they appear in unexpected ways in pure mathematics. It has been shown that Hilbert schemes $\mathrm{Hilb}^n(\mathbb{C}^3)$ of points in 3-space are critical loci of superpotentials. This description enables one to compute enumerative structures attached to Hilbert schemes. For example, one can derive the virtual motive $[\mathrm{Hilb}^n(\mathbb{C}^3)]$, the degree 0 Donaldson–Thomas invariants and partition functions [2], and the associated mixed Hodge module [3]. Unfortunately, this description does not carry over to $\mathrm{Hilb}^n(\mathbb{C}^d)$ when $d > 3$ and the Hilbert scheme is not smooth. This raises the question:

Question 0.8. *Do superpotentials exist for more general Hilbert schemes or moduli spaces, and can we explicitly compute them?*

Chris Brav (Higher School of Economics) and I have recently begun collaborating to find new theoretical and computational techniques that produce concrete superpotential equations in more general settings. Our aim is to use these superpotentials to study the topology and enumerative invariants of more general moduli spaces. Using *Macaulay2*, we have produced explicit superpotentials for the Hilbert schemes $\mathrm{Hilb}^n(\mathbb{C}^3)$, where $4 \leqslant n \leqslant 8$. A goal is to write a *Macaulay2* package that allows detailed exploration of concrete superpotentials. In particular, we are studying how A_∞-structures on Ext-algebras generate superpotentials. The study of even simple moduli spaces is generally difficult, so we believe this new approach will lead to new insights. Moreover, we have already used mathematical techniques touching on representation theory, homological algebra, and derived geometry, so this project promises to lead to rich new connections.

Connections with combinatorics. Polygons and polyhedra have been of primary mathematical interest since antiquity. Like atoms in physics, their simple composition means polytopes still underlie many modern mathematical concepts. Rational convex polytopes determine important toric varieties, for instance, which are further generalized by log schemes. Murphy's Law for Hilbert schemes is a consequence of Vershik–Mnëv's "universality" relating combinatorial arrangements to real algebraic sets. I have just begun working with Eran Nevo (The Hebrew University of Jerusalem) on understanding the exciting connections between complex geometry (Lefschetz theorems and toric geometry), commutative algebra (Hilbert series and K-polynomials), and combinatorics (the g-conjecture, Kruskal–Katona, polytope invariants). This area at the crossroads of geometry, algebra, and combinatorics promises to be a fertile ground for new discoveries. Understanding and discovering new connections between these subfields of mathematics will form a large component of my research in the near future.

References

[1] Paul S. Aspinwall and Lukasz M. Fidkowski. Superpotentials for quiver gauge theories. *J. High Energy Phys.*, 2006(10):047, 25, 2006.

[2] Kai Behrend, Jim Bryan, and Balázs Szendr˝ oi. Motivic degree zero Donaldson-Thomas invariants. *Invent. Math.*, 192(1):111–160, 2013.

[3] Alexandru Dimca and Balázs Szendr˝ oi. The Milnor fibre of the Pfaffian and the Hilbert scheme of four points on $\mathbb{C}^3$. *Math. Res. Lett.*, 17(2):243–262, 2010.

[4] Balin Fleming. *Arc schemes in logarithmic algebraic geometry*. PhD thesis, University of Michigan, 2015.

[5] Gerd Gotzmann. Some Irreducible Hilbert Schemes. *Math. Z.*, 201(1):13–17, 1989.

[6] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[7] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[8] Robin Hartshorne. Connectedness of the Hilbert scheme. *Inst. Hautes Études Sci. Publ. Math.*, 29:5–48, 1966.

[9] Kalle Karu and Andrew P. Staal. Singularities of log varieties via jet schemes, 2012. arXiv:1201.6646 [math.AG].

[10] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[11] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

[12] F. S. Macaulay. Some Properties of Enumeration in the Theory of Modular Systems. *Proc. London Math. Soc.*, S2-26(1):531, 1927.

[13] Mircea Mustaţă. Jet schemes of locally complete intersection canonical singularities. *Invent. Math.*, 145(3):397–424, 2001. With an appendix by David Eisenbud and Edward Frenkel.

[14] Mircea Mustaţă. Singularities of pairs via jet schemes. *J. Amer. Math. Soc.*, 15(3):599–615 (electronic), 2002.

[15] Ragni Piene and Michael Schlessinger. On the Hilbert scheme compactification of the space of twisted cubics. *Amer. J. Math.*, 107(4):761–774, 1985.

[16] Alyson Reeves and Mike Stillman. Smoothness of the lexicographic point. *J. Algebraic Geom.*, 6(2):235–246, 1997.

[17] Ed Segal. The A_∞ deformation theory of a point and the derived categories of local Calabi-Yaus. *J. Algebra*, 320(8):3232–3268, 2008.

[18] Andrew P. Staal. *Irreducibility of Random Hilbert Schemes*. PhD thesis, Queen's University, 2016.

[19] Andrew P. Staal. The Ubiquity of Smooth Hilbert Schemes, 2017. arXiv:1702.00080 [math.AG].

[20] Jan Stevens. Computing versal deformations. *Experiment. Math.*, 4(2):129–144, 1995.

[21] Ravi Vakil. Murphy's law in algebraic geometry: badly-behaved deformation spaces. *Invent. Math.*, 164(3):569–590, 2006.