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Research Statement
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Overview. My research is in algebraic geometry, where I work to understand the geometry of parameter spaces. Parameter spaces are geometric spaces that parametrize mathematical objects with common properties. Each point on a parameter space corresponds to some algebraic equation that we might wish to better understand. Various parameter spaces lie at the heart of algebraic geometry, from fundamental examples, like projective spaces and Grassmannians, to the esoteric, like Hilbert schemes, Quot schemes, and more general moduli spaces.

The parameter spaces at the core of my research are Hilbert schemes and jet schemes. Hilbert schemes parametrize subschemes in projective space with chosen algebro-geometric data captured by a Hilbert polynomial. My main research contributions, and my future research goals, lie in broadening our knowledge of the geography of Hilbert schemes, and finding connections between Hilbert schemes and other areas of mathematics. In my Ph.D. thesis [19], I formulate a framework organizing Hilbert schemes via two operations on Hilbert polynomials. I prove that to some extent this framework categorizes Hilbert schemes according to the complexity of their geometry, helping to identify the boundary between well-behaved and pathological Hilbert schemes. Using combinatorial commutative algebra and computational deformation theory, I am expanding the list of Hilbert schemes that we can explicitly understand. My earlier research broadens the applicability of jet schemes to the context of log geometry. Jet schemes are parameter spaces that serve as specialized tools, akin to tangent bundles, and shed light on singularities.

Doctoral Work – Towards the geography of Hilbert schemes. Defined in [4], Hilbert schemes are important parameter spaces, generalizing projective spaces and Grassmannians. Yet surprisingly little is known about their geometry in general. Hilbert schemes are connected [5], and Hilbert schemes always contain a nonsingular point [15], but beyond this general results are scarce. According to “Murphy’s Law” in algebraic geometry [21] there exist Hilbert schemes with arbitrarily complicated singularities and nonreduced structures. However, there is a gap in our knowledge of the geography of Hilbert schemes, between the tractable examples and the known pathologies. Basic geometric questions about Hilbert schemes—whether they are nonsingular, how many irreducible components they have, how these components intersect, why some have many components while others have few—are usually difficult to answer, and general statements are sparse, as are fully analyzed examples. These questions form the core motivation for my research, as I want to find and understand families of Hilbert schemes with simple geometry, to completely answer basic geometric questions for these spaces, and to probe the transition into chaos.

In [19], I prove that random Hilbert schemes are smooth and irreducible with probability greater than 0.5. In [18], I improve this with a classification of Hilbert schemes having unique Borel-fixed points. In particular, I utilize a tree structure on the set of all Hilbert schemes, which lets us interpret the set as having probability distributions with infinite support. I relate the tree structure to the generation of Borel-fixed points, which serves as a proxy for complexity in the geometry Hilbert schemes, and we find that simple Hilbert schemes are rather dense in the graph.

This research stems from experimental computations of the local geometry of specific Hilbert schemes, in the spirit of well-known examples [14, 20]. Ideally, we would have a suite of methods that, given any Hilbert scheme, return the number of its irreducible components, their dimensions, how they intersect, whether they are rational, etc. This goal is unreasonably optimistic, so rather than aim directly at it, my approach works as follows. A combinatorial classification of polynomials $\mathsf{p}(t) \in \mathbb{Q}[t]$ corresponding to nonempty Hilbert schemes by expressions of the form $\sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$, where $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$ are integers, is given in [10]. I identify an infinite family of binary trees, whose nodes are all Hilbert schemes, organized according to their Hilbert polynomials. The edges connecting a parent node to its children are defined by two operations $\Psi\colon \mathsf{p} \mapsto \mathsf{p}+1$ and $\Phi\colon \mathsf{p} \mapsto \sum_{j=1}^{r} \binom{t+b_j+1-(j-1)}{b_j+1}$. Patterns emerge in this structure reflecting the complexity of the geometry of Hilbert schemes.

Borel-fixed points mark interesting geometry on Hilbert schemes. Well-known work exploits their combinatorics to show that Hilbert schemes are connected [5], have bounded radii [16], and always have smooth points [15]. Using *Macaulay2* [3] to implement an algorithm that generates Borel-fixed ideals, and running the implementation through the binary trees, I discovered that ascending through the trees via Φ reduces complexity in the geometry of the corresponding Hilbert schemes, while Ψ increases complexity. I prove in [19] that if the last edge to the node of a Hilbert scheme is Φ, then the Hilbert scheme has a unique Borel-fixed ideal. Such a Hilbert scheme must then be irreducible and nonsingular; [2] also shows that these Hilbert schemes are irreducible, by different methods. I extend this in [18] to a classification of all Hilbert schemes with unique Borel-fixed points by conditions on the numbers $b_1, b_2, \ldots, b_r$. These infinite families comprise over one-half of all Hilbert schemes, in the context of these binary trees. Such density is a surprising counterpoint to the current wisdom that general Hilbert schemes become arbitrarily complicated.

The main idea of the proof is to understand how moving through the trees relates to an algorithm [17, 11] that produces Borel-fixed ideals. The map Φ corresponds to lifting lists of Borel-fixed ideals to larger rings. Lexicographic ideals lift to obtain minimal Hilbert polynomials, and we prove that nonlexicographic Borel-fixed ideals lift to obtain nonminimal Hilbert polynomials. We achieve this via detailed study of Hilbert series of Borel-fixed ideals. In particular, tracking the degrees and maximum indices of minimal monomial generators shows that the degrees of K-polynomials of lexicographic ideals are always strictly greater than the degrees of K-polynomials of corresponding nonlexicographic Borel-fixed ideals. That the Hilbert schemes in [2] have unique Borel-fixed points is a corollary of this result. Bootstrapping this further leads to a classification of all Hilbert schemes with unique Borel-fixed ideals, by using the binary trees to bound the numbers of degrees in which distinct minimal monomial generators of lexicographic ideals may appear.

I believe that more information about the geometry of Hilbert schemes remains to be discovered in these binary trees, and I propose to continue studying this structure in future research. For instance, it appears to often be the case that if the last two edges on the path of a Hilbert scheme are Φ followed by Ψ then the number of Borel-fixed ideals on the target Hilbert scheme is small, and moreover does not change if the last Φ is replaced by a positive power Φ^a. Theses schemes are good candidates for expanding our list of tractable examples, and I wish to know whether these Hilbert schemes are irreducible, rational, have restricted singularity types, etc. Several other questions arise—Is there a minimal number of Ψ edges, after the last Φ, such that we are guaranteed to produce multiple irreducible components? Can we refine our understanding of radii of Hilbert schemes via their paths in these binary trees? What other relationships exist between these paths and their numbers of Borel-fixed ideals, or the geometric complexity of Hilbert schemes? If we bound the number of Ψ edges, can we say anything interesting about the types of singularities that appear on Hilbert schemes? I also wish to see rigorous analysis via local deformation-theoretic calculations of concrete examples, using the binary trees as a guide. Further, can we generalize any of the above to multigraded/toric/tropical Hilbert schemes, or other moduli? I envision all of these questions as trailheads to paths of future research.

Masters Work – Log-jet schemes and singularities. This research aims to apply jet schemes in log geometry. Jets are higher-degree versions of tangent vectors—infinitesimal bits of curves approximating other spaces. Jet schemes are analogues of tangent bundles, parametrizing all jets on a fixed base space. Jet schemes have fruitful applications to the study of singularities, as their geometry reflects the singularities of the base; for example, all jet schemes of a local complete intersection variety are irreducible if and only if the variety has canonical singularities [12]. Singularities of pairs, and invariants like the log-canonical threshold, also have geometric manifestations in jet schemes [13]. Many of these properties of jet schemes over singularities are proved using motivic integration, which is a type of integration theory on limits of spaces of jets [9].

As higher dimensional geometry and the minimal model program developed, singularities of pairs became more than just a technical generalization of singularities of varieties, or singularities of divisors in smooth varieties. However, [8] mentions that researchers were vague about what the morphisms in a suitable category should be. Log geometry, developed in [7] for different purposes, rigorously defines a category for related, but distinct, types of pairings. This prompts the question of whether log geometry can supply further insight into higher dimensional singularities.

Kalle Karu and I initiate a program [6] towards understanding singularities of pairs in the category of log schemes. We define local complete intersection varieties and canonical singularities in the log category, and relate canonical log varieties with canonical and log-canonical pairs, comparing equations of log jet schemes with those of regular jet schemes. The main result of this paper proves a log geometry analogue of the main result in [12]—that every log jet scheme of a local complete intersection log variety is irreducible if and only if the log variety is canonical. This paper demonstrates that the category of log schemes is a viable setting for studying the singularities of higher dimensional geometry, and that a key tool for studying singularities of pairs carries over. New progress has been made in [1] towards further understanding how log jet schemes relate to singularities and generalizing motivic integration to the log category.

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