# THE UBIQUITY OF SMOOTH HILBERT SCHEMES

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ABSTRACT. We investigate the geography of Hilbert schemes that parametrize closed subschemes of projective space with a specified Hilbert polynomial. We classify Hilbert schemes with unique Borel-fixed points via combinatorial expressions for their Hilbert polynomials. We realize the set of all nonempty Hilbert schemes as a probability space and prove that Hilbert schemes are irreducible and nonsingular with probability greater than $0.5$.

## 1. INTRODUCTION

Hilbert schemes parametrizing closed subschemes with a fixed Hilbert polynomial in projective space are fundamental moduli spaces. With the exception of Hilbert schemes parametrizing hypersurfaces [ACG11, Example 2.3] and points in the plane [Fog68], the geometric features of typical Hilbert schemes are still poorly understood. Techniques for producing pathological Hilbert schemes are known, generating Hilbert schemes with many irreducible components [Iar72, FP96], with generically nonreduced components [Mum62], and with arbitrary singularity types [Vak06]. What should we expect from a random Hilbert scheme? Can we understand the geography of Hilbert schemes? Our answer is that the set of nonempty Hilbert schemes forms a graph and a discrete probability space, and that irreducible, nonsingular Hilbert schemes are unexpectedly common.

Let $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ denote the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $\mathsf{p}$. Polynomials that are Hilbert polynomials of homogeneous ideals are classified in [Mac27]. Any such admissible Hilbert polynomial $\mathsf{p}(t)$ has a combinatorial expression $\sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$ for $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$. Our first main result is the following.

**Theorem 1.1.** *The lexicographic ideal is the unique saturated Borel ideal of codimension $c$ with Hilbert polynomial $\mathsf{p}$ if and only if:*

*(i) $c \geq 2$ and either $b_r > 0$ or $r \leq 2$; or*
*(ii) $c = 1$ and either $b_r > 0$, $b_1 = b_r$, or $r - s \leq 2$, where $b_1 = b_2 = \cdots = b_s > b_{s+1} \geq \cdots \geq b_r$.*

Borel ideals generalize lexicographic ideals and in characteristic $0$ define Borel-fixed points on Hilbert schemes. Some basic general properties of Hilbert schemes have been extracted from these ideals. Rational curves linking Borel-fixed points prove connectedness in [Har66, PS05]. The thesis [Bay82] uses them to give equations for Hilbert schemes and proposes studying their tangent cones. Further, [Ree95] studies their combinatorial properties to give general bounds for radii of Hilbert schemes, and [RS97] proves that lexicographic points are nonsingular. Theorem 1.1 specifies an explicit collection of well-behaved Hilbert schemes, generalizes the main result of [Got89], and improves our understanding of the geography of Hilbert schemes.

This collection of Hilbert schemes is ubiquitous. Our new interpretation of Macaulay's classification identifies an infinite binary tree $\mathcal{H}_c$ whose vertices are the Hilbert schemes

$\text{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ parametrizing codimension $c = n - \deg \mathsf{p}$ subschemes, for each positive $c \in \mathbb{Z}$. Assuming that vertices at a fixed height are equally likely, combining probability distributions for the height with a distribution for the parametrized codimension $c$ endows the set of Hilbert schemes with the structure of a discrete probability space. This leads to our second main result.

**Theorem 1.2.** *The probability that a random Hilbert scheme is irreducible and nonsingular is greater than* $0.5$*.*

This theorem counterintuitively suggests that the geometry of the majority of Hilbert schemes is understandable. To prove Theorems 1.1 and 1.2, we study the algorithm generating saturated Borel ideals first described in [Ree92] and later generalized in [Moo12, CLMR11]. We obtain precise information about Hilbert series and $K$-polynomials of saturated Borel ideals. The primary technical result we need is the following.

**Theorem 1.3.** *Let* $I \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$ *be a saturated Borel ideal with Hilbert polynomial* $\mathsf{p}$*, let* $L_n^{\mathsf{p}}$ *be the corresponding lexicographic ideal in* $\mathbb{K}[x_0, x_1, \ldots, x_n]$*, and let* $\mathsf{K}_I$ *be the numerator of the Hilbert series of* $I$*. If* $I \neq L_n^{\mathsf{p}}$*, then we have* $\deg \mathsf{K}_I < \deg \mathsf{K}_{L_n^{\mathsf{p}}}$*.*

The structure of the paper is as follows. In Section 2, we introduce two binary relations on the set of admissible Hilbert polynomials and show that they generate all such polynomials; see Theorem 2.10. The set of lexicographic ideals is then partitioned by codimension into infinitely many binary trees in Section 3. Geometrically, these are trees of Hilbert schemes, as every Hilbert scheme contains a unique lexicographic ideal. Section 4 makes explicit this graph-theoretical structure on the set of Hilbert schemes. To identify a sufficiently dense family of irreducible, nonsingular Hilbert schemes, we review saturated Borel ideals in Section 5 and we examine their $K$-polynomials in Section 6. The main results are in Section 7.

**Conventions.** Throughout, $\mathbb{K}$ is an algebraically closed field, $\mathbb{N}$ is the set of nonnegative integers, and $\mathbb{K}[x_0, x_1, \ldots, x_n]$ is the standard $\mathbb{Z}$-graded polynomial ring. The Hilbert function, polynomial, series, and $K$-polynomial of the quotient $\mathbb{K}[x_0, x_1, \ldots, x_n]/I$ by a homogeneous ideal $I$ are denoted $\mathsf{h}_I, \mathsf{p}_I, \mathsf{H}_I$, and $\mathsf{K}_I$, respectively.

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## 2. The Tree of Admissible Hilbert Polynomials

We identify a graph structure on the set of numerical polynomials determining nonempty Hilbert schemes of projective spaces. The pioneering work [Mac27] gives a combinatorial classification of these polynomials. We introduce two binary relations to equip this set with the structure of an infinite binary tree.

Let $\mathbb{K}$ be an algebraically closed field and let $\mathbb{K}[x_0, x_1, \ldots, x_n]$ denote the homogeneous (standard $\mathbb{Z}$-graded) coordinate ring of $n$-dimensional projective space $\mathbb{P}^n$. Let $M$ be a finitely generated graded $\mathbb{K}[x_0, x_1, \ldots, x_n]$-module. The ***Hilbert function*** $\mathsf{h}_M \colon \mathbb{Z} \to \mathbb{Z}$ of $M$

is defined by $\mathsf{h}_M(i) := \dim_{\mathbb{K}}(M_i)$ for all $i \in \mathbb{Z}$. Every such $M$ has a ***Hilbert polynomial*** $\mathsf{p}_M$, that is, a polynomial $\mathsf{p}_M(t) \in \mathbb{Q}[t]$ such that $\mathsf{h}_M(i) = \mathsf{p}_M(i)$ for $i \gg 0$; see [BH93, Theorem 4.1.3]. For a homogeneous ideal $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$, let $\mathsf{h}_I$ and $\mathsf{p}_I$ denote the Hilbert function and Hilbert polynomial of the quotient module $\mathbb{K}[x_0, x_1, \dots, x_n]/I$, respectively. If $X \subseteq \mathbb{P}^n$ is a nonempty closed subscheme, then there is a unique saturated homogeneous ideal $I_X \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$ such that $X = \operatorname{Proj}\big(\mathbb{K}[x_0, x_1, \dots, x_n]/I_X\big)$; see [Har77, Corollary II.5.16]. We define the ***Hilbert function*** $\mathsf{h}_X$ of $X$ to be the Hilbert function $\mathsf{h}_{I_X} = \mathsf{h}_{\mathbb{K}[x_0,x_1,\dots,x_n]/I_X}$, and the ***Hilbert polynomial*** $\mathsf{p}_X$ of $X$ to be $\mathsf{p}_{I_X} = \mathsf{p}_{\mathbb{K}[x_0,x_1,\dots,x_n]/I_X}$.

As a first example, we describe the Hilbert polynomial of $\mathbb{P}^n$.

**Example 2.1.** Fix a nonnegative integer $n \in \mathbb{N}$. The stars-and-bars argument [Sta12, Section 1.2] shows that the number of independent homogeneous polynomials of degree $i \in \mathbb{Z}$ in $n+1$ variables is $\binom{n+i}{n}$ and equals $0$ for $i < 0$. That is, we have $\mathsf{h}_S(i) = \binom{n+i}{n}$ for $S := \mathbb{K}[x_0, x_1, \dots, x_n]$. The equality $\mathsf{h}_S(i) = \mathsf{p}_S(i)$ is only valid for $i \geq -n$, because $\mathsf{p}_S$ only has roots $-n, -(n-1), \dots, -1$, whereas $\mathsf{h}_S(i) = 0$ for all $i < 0$.

**Remark 2.2.** We often treat binomial coefficients as polynomials. Following [GKP94, Section 5.1], for a variable $t$ and $a, b \in \mathbb{Z}$, let $\binom{t+a}{b} := \frac{(t+a)(t+a-1)\cdots(t+a-b+1)}{b!} \in \mathbb{Q}[t]$ if $b \geq 0$, and $\binom{t+a}{b} := 0$ otherwise. If $b \geq 0$, then $\binom{t+a}{b}$ has degree $b$ in $t$, with zeros $-a, -(a-1), \dots, -(a-b+1)$, so that $\binom{t+a}{b}|_{t=j} \neq \binom{j+a}{b} = 0$ for $j < -a$. Interestingly, [Mac27, p. 533] uses distinct notation for polynomial and integer binomial coefficients.

A polynomial is an ***admissible*** Hilbert polynomial if it is the Hilbert polynomial of a nonempty closed subscheme of a projective space. As a consequence, admissible Hilbert polynomials correspond to nonempty Hilbert schemes. We have the following classification.

**Proposition 2.3.** *The following conditions are equivalent:*

*(i) The polynomial $\mathsf{p}(t) \in \mathbb{Q}[t]$ is an admissible Hilbert polynomial.*

*(ii) There exist integers $e_0 \geq e_1 \geq \cdots \geq e_d > 0$ such that $\mathsf{p}(t) = \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$.*

*(iii) There exist integers $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$ such that $\mathsf{p}(t) = \sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$.*

*Moreover, the correspondences between admissible Hilbert polynomials and sequences of $e_i$'s, or of $b_j$'s, are bijective.*

*Proof.*

(i) ⇔ (ii) This is proved in [Mac27, Part I]; see the formula for "$\chi(\ell)$" at the bottom of p. 536. For a modern account, see [Har66, Corollary 3.3 and Corollary 5.7].

(i) ⇔ (iii) This follows from [Got78, Erinnerung 2.4]; see also [BH93, Exercise 4.2.17].

The uniqueness of the sequences of integers attached to an admissible polynomial is also explained by the aforementioned sources. □

We define the ***Macaulay–Hartshorne expression*** of an admissible Hilbert polynomial $\mathsf{p}$ to be its expression $\mathsf{p}(t) = \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$, for $e_0 \geq e_1 \geq \cdots \geq e_d > 0$. Similarly, we define the ***Gotzmann expression*** of $\mathsf{p}$ to be its expression $\mathsf{p}(t) = \sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$, for $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$. From these, we see the degree $d = b_1$, the leading coefficient $e_d/d!$, and the ***Gotzmann number*** $r$ of $\mathsf{p}$, which bounds the Castelnuovo–Mumford regularity of saturated ideals with Hilbert polynomial $\mathsf{p}$; see [IK99, Definition C.12].

Macaulay–Hartshorne and Gotzmann expressions are conjugate. The ***conjugate*** partition to a partition $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_k)$ of an integer $\ell = \sum_{i=1}^{k} \lambda_i$ is the partition of $\ell$ obtained from the Ferrers diagram of $\lambda$ by interchanging rows and columns, having $\lambda_i - \lambda_{i+1}$ parts equal to $i$; see [Sta12, Section 1.8].

**Lemma 2.4.** *If $\mathsf{p}(t) \in \mathbb{Q}[t]$ is an admissible Hilbert polynomial with Macaulay–Hartshorne expression $\sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$, for $e_0 \geq e_1 \geq \cdots \geq e_d > 0$, and Gotzmann expression $\sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$, for $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$, then $r = e_0$ and the nonnegative partition $(b_1, b_2, \ldots, b_r)$ is conjugate to the partition $(e_1, e_2, \ldots, e_d)$.*

*Proof.* Rewriting the Macaulay–Hartshorne expression of $\mathsf{p}$ as

$$\sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} + \sum_{i=0}^{d-1} \binom{t+i-e_d}{i+1} - \binom{t+i-e_i}{i+1},$$

we prove that $\sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \sum_{j=1}^{e_d} \binom{t+d-(j-1)}{d}$. If $d = 0$, then $\binom{t}{1} - \binom{t-e_0}{1} = e_0$ holds. If $d > 0$, then we have $\sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \left[\sum_{i=0}^{d-1} \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1}\right] + \binom{t+d}{d+1} - \binom{t+d-e_d}{d+1}$. By induction, this equals $\left[\sum_{j=1}^{e_d} \binom{t+(d-1)-(j-1)}{d-1}\right] + \binom{t+d}{d+1} - \binom{t+d-e_d}{d+1}$. The addition formula [GKP94, Section 5.1] yields

$$\binom{t+d}{d+1} - \binom{t+d-e_d}{d+1} = \left[\sum_{j=2}^{e_d} \binom{t+d-(j-1)}{d}\right] + \binom{t+d-e_d}{d} \text{ and}$$

$$\binom{t+d}{d} = \left[\sum_{j=1}^{e_d} \binom{t+(d-1)-(j-1)}{d-1}\right] + \binom{t+d-e_d}{d},$$

and we obtain $\sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \sum_{j=1}^{e_d} \binom{t+d-(j-1)}{d}$, as desired.

Now we write the Macaulay–Hartshorne expression as

$$\mathsf{p}(t) = \sum_{j=1}^{e_d} \binom{t+d-(j-1)}{d} + \left[\sum_{i=0}^{d-1} \binom{s+i}{i+1} - \binom{s+i-(e_i-e_d)}{i+1}\right]_{s:=t-e_d}$$

and repeat the decomposition on the second part. With $s := t - e_d$, this yields the sum

$$\left[\sum_{i=0}^{d-1} \binom{s+i}{i+1} - \binom{s+i-(e_{d-1}-e_d)}{i+1} + \sum_{i=0}^{d-2} \binom{s+i-(e_{d-1}-e_d)}{i+1} - \binom{s+i-(e_i-e_d)}{i+1}\right],$$

whose first part equals $\sum_{k=1}^{e_{d-1}-e_d} \binom{s+(d-1)-(k-1)}{d-1}$, by the previous paragraph. Reindexing with $j := k + e_d$ and evaluating at $s := t - e_d$ gives $\sum_{j=e_d+1}^{e_{d-1}} \binom{t+(d-1)-(j-1)}{d-1}$. Therefore, we have $\sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$ when $b_j = i$ for all $e_i \geq j > e_{i+1}$, where $e_{d+1} := 0$. This shows that $r = e_0$ and that $e_i - e_{i+1}$ parts equal $i$ in the partition associated to the Gotzmann expression of $\mathsf{p}$, for all $0 \leq i \leq d$. Finally, $\sum_{j=1}^{r} b_j = \sum_{i=0}^{d} (e_i - e_{i+1}) i = \sum_{i=1}^{d} e_i$ holds and it follows that $(b_1, b_2, \ldots, b_r)$ is conjugate to $(e_1, e_2, \ldots, e_d)$. ∎

We define the ***Macaulay–Hartshorne partition*** of an admissible Hilbert polynomial $\mathsf{p}(t) = \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$ to be the partition $(e_0, e_1, \ldots, e_d)$, and the ***Gotzmann partition*** of $\mathsf{p}(t) = \sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$ to be the nonnegative partition $(b_1, b_2, \ldots, b_r)$.

**Example 2.5.** The twisted cubic curve $X \subset \mathbb{P}^3$ is defined by the ideal $I_X \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ of 2-minors of the matrix $\left[\begin{smallmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{smallmatrix}\right]$. The Macaulay–Hartshorne and Gotzmann expressions for the Hilbert polynomial of the twisted cubic are

$$\begin{aligned} \mathsf{p}_X(t) = 3t+1 &= \left[\binom{t+0}{0+1} - \binom{t+0-4}{0+1}\right] + \left[\binom{t+1}{1+1} - \binom{t+1-3}{1+1}\right] \\ &= \binom{t+1}{1} + \binom{t+1-1}{1} + \binom{t+1-2}{1} + \binom{t+0-3}{0}, \end{aligned}$$

respectively. The partitions are $(e_0, e_1) = (4, 3)$ and $(b_1, b_2, b_3, b_4) = (1, 1, 1, 0)$. Observe that $(3)$ is conjugate to $(1, 1, 1)$, but that $(1, 1, 1, 0)$ has $r = e_0 = 4$.

We describe two binary relations on the set of admissible Hilbert polynomials. Let $\mathsf{p}$ be an admissible Hilbert polynomial with Macaulay–Hartshorne partition $(e_0, e_1, \dots, e_d)$ and Gotzmann partition $(b_1, b_2, \dots, b_r)$. We define a mapping $\Phi$, from the set of admissible Hilbert polynomials to itself, that takes $\mathsf{p}$ to the polynomial $\Phi(\mathsf{p})$ with Macaulay–Hartshorne partition $(e_0, e_0, e_1, \dots, e_d)$ and Gotzmann partition $(b_1+1, b_2+1, \dots, b_r+1)$. Explicitly, we have $[\Phi(\mathsf{p})](t) = \binom{t+0}{0+1} - \binom{t+0-e_0}{0+1} + \sum_{i=1}^{d+1} \binom{t+i}{i+1} - \binom{t+i-e_{i-1}}{i+1} = \sum_{j=1}^{r} \binom{t+(b_j+1)-(j-1)}{b_j+1}$, which is admissible, by Proposition 2.3.

To define the second binary relation on the set of admissible Hilbert polynomials, let $\Psi \colon \mathbb{Q}[t] \to \mathbb{Q}[t]$ be the mapping taking a polynomial $\mathsf{p}$ to $1+\mathsf{p}$. If $\mathsf{p}(t)$ is an admissible Hilbert polynomial with Macaulay–Hartshorne partition $(e_0, e_1, \dots, e_d)$ and Gotzmann partition $(b_1, b_2, \dots, b_r)$, then $\Psi(\mathsf{p}) = 1 + \mathsf{p}$ is also an admissible Hilbert polynomial, with Macaulay–Hartshorne partition $(e_0+1, e_1, e_2, \dots, e_d)$ and Gotzmann partition $(b_1, b_2, \dots, b_r, 0)$. We have $[\Psi(\mathsf{p})](t) = \binom{t+0}{0+1} - \binom{t+0-(e_0+1)}{0+1} + \sum_{i=1}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1} = \sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j} + \binom{t+0-r}{0}$. Therefore, the restriction of $\Psi$ defines a mapping from the set of admissible Hilbert polynomials to itself, which we also denote by $\Psi$.

**Example 2.6.** The simplest admissible Hilbert polynomial is that of a reduced point in projective space, with Gotzmann expression $1 = \binom{t+0-0}{0}$. We have $\Phi(1) = \binom{t+1-0}{1} = t+1$, which is the Hilbert polynomial of a reduced line. Also, $\Psi(1) = \binom{t+0-0}{0} + \binom{t+0-1}{0} = 2$ is the Hilbert polynomial of two points in projective space.

To better understand $\Phi$, we consider the ***backwards difference operator*** $\nabla \colon \mathbb{Q}[t] \to \mathbb{Q}[t]$, defined by $\mathsf{q} \mapsto [\nabla(q)](t) := q(t) - q(t-1)$; compare with [BH93, Lemma 4.1.2]. We collect elementary properties showing the interplay between $\Psi$, $\Phi$, and $\nabla$. Backwards differences are discrete derivatives, and Lemma 2.7(ii) shows that $\Phi$ gives indefinite sums of admissible polynomials. Part (iii) is a well-known discrete analogue of the Fundamental Theorem of Calculus.

**Lemma 2.7.** *If $\mathsf{p}(t)$ is an admissible Hilbert polynomial with Macaulay–Hartshorne partition $(e_0, e_1, \dots, e_d)$ and Gotzmann partition $(b_1, b_2, \dots, b_r)$, then the following hold:*

*(i)* $[\nabla(\mathsf{p})](t) = \sum_{j=1}^{r} \binom{t+(b_j-1)-(j-1)}{b_j-1} = \sum_{i=0}^{d-1} \binom{t+i}{i+1} - \binom{t+i-e_{i+1}}{i+1}$;
*(ii)* $\nabla\, \Psi^a\, \Phi(\mathsf{p}) = \mathsf{p}$*, for all* $a \in \mathbb{N}$;
*(iii) if* $\deg \mathsf{p} > 0$ *and* $k \in \{1, 2, \dots, r\}$ *is the largest index such that* $b_k \neq 0$*, then we have* $\mathsf{p} - \Phi\, \nabla(\mathsf{p}) = r - k$*, but if* $\deg \mathsf{p} = 0$*, then* $\nabla(\mathsf{p}) = 0$*; and*
*(iv)* $[(\Phi\, \Psi - \Psi\, \Phi)(\mathsf{p})](t) = t - r$.

**Remark 2.8.** Setting $a = 0$ in Part (ii) shows that $\nabla\,\Phi(\mathsf{p}) = \mathsf{p}$, so that Part (iii) shows that $(\nabla\,\Phi - \Phi\,\nabla)(\mathsf{p}) = r - k$. In other words, applying $\Phi$ and then $\nabla$ returns $\mathsf{p}$, but applying $\nabla$ and then $\Phi$ may alter the constant term of $\mathsf{p}$.

*Proof.*

(i) Because $\nabla$ is a linear operator on $\mathbb{Q}[t]$, it suffices to prove the statement for polynomials of the form $\binom{t+b-i}{b}$, for $b, i \in \mathbb{N}$. By definition of $\nabla$ and the addition formula, we have $[\nabla(\mathsf{p})]\,(t) = \binom{t+b-i}{b} - \binom{t-1+b-i}{b} = \binom{t+b-1-i}{b-1}$.

(ii) We have $[\nabla\,\Phi(\mathsf{p})]\,(t) = \nabla\left(\sum_{j=1}^{r}\binom{t+b_j+1-(j-1)}{b_j+1}\right) = \sum_{j=1}^{r}\binom{t+b_j-(j-1)}{b_j} = \mathsf{p}(t)$, by Part (i). Further, $\nabla\,\Psi^a\,\Phi(\mathsf{p}) = \nabla\big(a + \Phi(\mathsf{p})\big)$, which equals $\nabla\,\Phi(\mathsf{p}) = \mathsf{p}$.

(iii) Because $\deg \mathsf{p} > 0$, there is a largest index $k \in \{1, \dots, r\}$ such that $b_k \neq 0$. Thus, $[\Phi\,\nabla(\mathsf{p})]\,(t) = \Phi\left(\sum_{j=1}^{k}\binom{t+b_j-1-(j-1)}{b_j-1}\right) = \sum_{j=1}^{k}\binom{t+b_j-(j-1)}{b_j}$. Hence, terms of the form $\binom{t+0-(j-1)}{0}$ are dropped, and as they all equal $1$, we are left with $\mathsf{p} - \Phi\,\nabla(\mathsf{p}) = r - k$.

(iv) We have $[\Phi\,\Psi(\mathsf{p})]\,(t) = \sum_{j=1}^{r+1}\binom{t+b_j+1-(j-1)}{b_j+1}$, where $b_{r+1} := 0$. On the other hand, we also have $[\Psi\,\Phi(\mathsf{p})]\,(t) = \sum_{j=1}^{r}\binom{t+b_j+1-(j-1)}{b_j+1} + \binom{t+0-r}{0}$, and taking the difference yields the polynomial $\big[\big(\Phi\,\Psi - \Psi\,\Phi\big)(\mathsf{p})\big]\,(t) = (t+1-r) - 1 = t - r$. □

**Example 2.9.** Example 2.5 shows that $\mathsf{p}_X(t) = \binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} + \binom{t-3}{0}$ for the twisted cubic curve $X \subset \mathbb{P}^3$, so we have $[\Phi(\mathsf{p}_X)]\,(t) = \binom{t+2}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t-2}{1} = \frac{3}{2}t^2 + \frac{5}{2}t - 1$ and $[\nabla\,\Phi(\mathsf{p}_X)]\,(t) = \nabla\left(\frac{3}{2}t^2 + \frac{5}{2}t - 1\right) = 3t + 1 = \mathsf{p}_X$. In the other order, we find that $[\nabla(\mathsf{p}_X)]\,(t) = \binom{t}{0} + \binom{t-1}{0} + \binom{t-2}{0} = 3$ and $[\Phi\,\nabla(\mathsf{p}_X)]\,(t) = \binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} = 3t$.

If we let $\mathsf{q}(t) := \frac{3}{2}t^2 + \frac{5}{2}t + 1$, then we also obtain $[\nabla(\mathsf{q})]\,(t) = 3t+1$. In fact, the expression $\mathsf{q}(t) = \left[\Psi^2\,\Phi(\mathsf{p}_X)\right](t) = \binom{t+2}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t-2}{1} + \binom{t-4}{0} + \binom{t-5}{0}$ shows that the polynomial $\mathsf{q}$ is an admissible Hilbert polynomial. This polynomial is the Hilbert polynomial of the (minimally embedded) first Hirzebruch surface, also known as the blow-up of $\mathbb{P}^2$ at a point.

The mappings $\Phi$ and $\Psi$ endow the set of admissible Hilbert polynomials with the structure of a graph.

**Theorem 2.10.** *The graph whose vertices correspond to admissible Hilbert polynomials and whose edges correspond to pairs of the form $\big(\mathsf{p}, \Psi(\mathsf{p})\big)$ and $\big(\mathsf{p}, \Phi(\mathsf{p})\big)$, for all admissible Hilbert polynomials $\mathsf{p}$, forms an infinite binary tree. Moreover, the root of the tree corresponds to the constant polynomial $1$.*

The infinite binary tree has $2^j$ vertices at height $j$, for all $j \in \mathbb{N}$. We define the ***Macaulay tree*** $\mathcal{M}$ to be the infinite binary tree of admissible Hilbert polynomials in Theorem 2.10.

*Proof.* We prove that the admissible Hilbert polynomial $\mathsf{p}(t) = \sum_{i=0}^{d}\binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$ equals

$$\mathsf{p} = \Psi^{e_0-e_1}\,\Phi\,\Psi^{e_1-e_2}\,\Phi \cdots \Phi\,\Psi^{e_{d-1}-e_d}\,\Phi\,\Psi^{e_d-1}(1),$$

where $e_0 \geq e_1 \geq \cdots \geq e_d > 0$. We proceed by induction on the length $d+1$ of the partition $(e_0, e_1, \cdots, e_d)$. If $d = 0$, then we have $\binom{t+0}{0+1} - \binom{t+0-e_0}{0+1} = e_0 = \Psi^{e_0-1}(1)$, which proves the claim. The induction hypothesis on the partition $(e_1, e_2, \cdots, e_d)$, shows that $\sum_{i=0}^{d-1}\binom{t+i}{i+1} - \binom{t+i-e_{i+1}}{i+1} = \Psi^{e_1-e_2}\,\Phi\,\Psi^{e_2-e_3}\,\Phi \cdots \Phi\,\Psi^{e_{d-1}-e_d}\,\Phi\,\Psi^{e_d-1}(1)$. Applying $\Phi$ to both sides, we obtain $e_1 + \sum_{i=1}^{d}\binom{t+i}{i+1} - \binom{t+i-e_i}{i+1} = \Phi\,\Psi^{e_1-e_2}\,\Phi\,\Psi^{e_2-e_3}\,\Phi \cdots \Phi\,\Psi^{e_{d-1}-e_d}\,\Phi\,\Psi^{e_d-1}(1)$ and applying $\Psi^{e_0-e_1}$ to both sides yields the desired equality. Because every finite binary sequence of $\Psi$'s and $\Phi$'s has a unique such expression, we obtain the result. □

A portion of $\mathcal{M}$ is displayed in Figure 1, in terms of Gotzmann expressions.

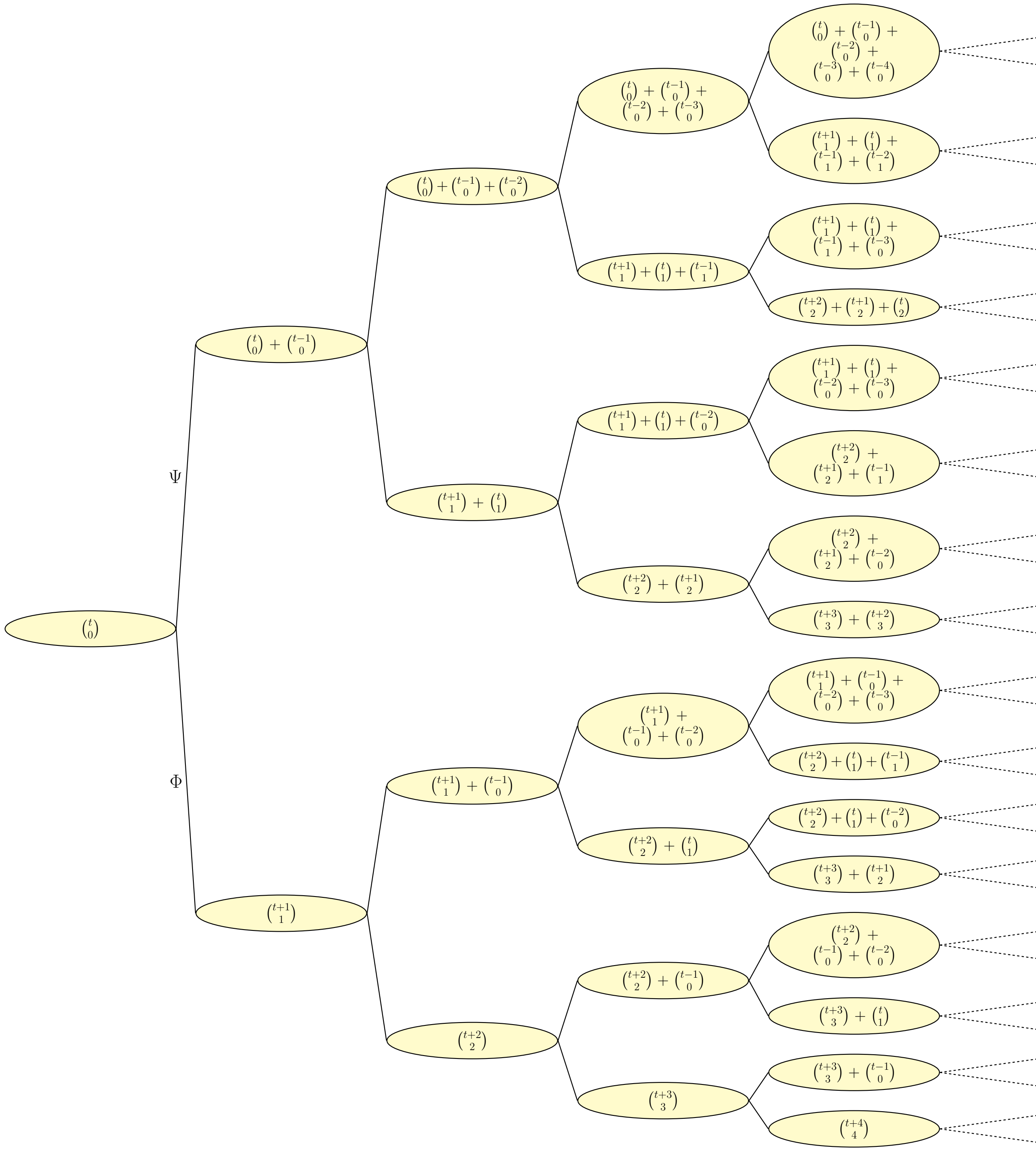


FIGURE 1. The Macaulay tree $\mathcal{M}$ to height 4 with Gotzmann expressions

**Remark 2.11.** The path from the root 1 of the tree $\mathcal{M}$ to an admissible Hilbert polynomial $\mathsf{p}(t) := \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1} = \sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$ is also encoded in the Gotzmann expression. In particular, we have the conjugate version

$$\mathsf{p} = \Phi^{b_r}\,\Psi\,\Phi^{b_{r-1}-b_r}\,\Psi \cdots \Psi\,\Phi^{b_2-b_3}\,\Psi\,\Phi^{b_1-b_2}(1)$$

of the expression in the proof of Theorem 2.10, where $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$. Moreover, these explicit expressions for the path from $1$ to $\mathsf{p}(t)$ show that the height of the vertex $\mathsf{p}(t)$ is $e_0 + d - 1 = r + b_1 - 1$.

**Example 2.12.** The Hilbert polynomial of the twisted cubic curve $X \subset \mathbb{P}^3$ has partitions $(e_0, e_1) = (4, 3)$ and $(b_1, b_2, b_3, b_4) = (1, 1, 1, 0)$; see Example 2.5. The Macaulay–Hartshorne expression and the proof of Theorem 2.10 give $\Psi^{4-3}\,\Phi\,\Psi^{3-1}(1) = 3t{+}1$, while the Gotzmann expression and Remark 2.11 give $\Phi^0\,\Psi\,\Phi^{1-0}\,\Psi\,\Phi^{1-1}\,\Psi\,\Phi^{1-1}(1) = 3t + 1$. This path is shown in Figure 2. From Example 2.9, we find that the path associated to the Hilbert polynomial of the minimally embedded Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \subset \mathbb{P}^4$ is $\Psi^2\,\Phi\,\Psi\,\Phi\,\Psi^2(1)$.

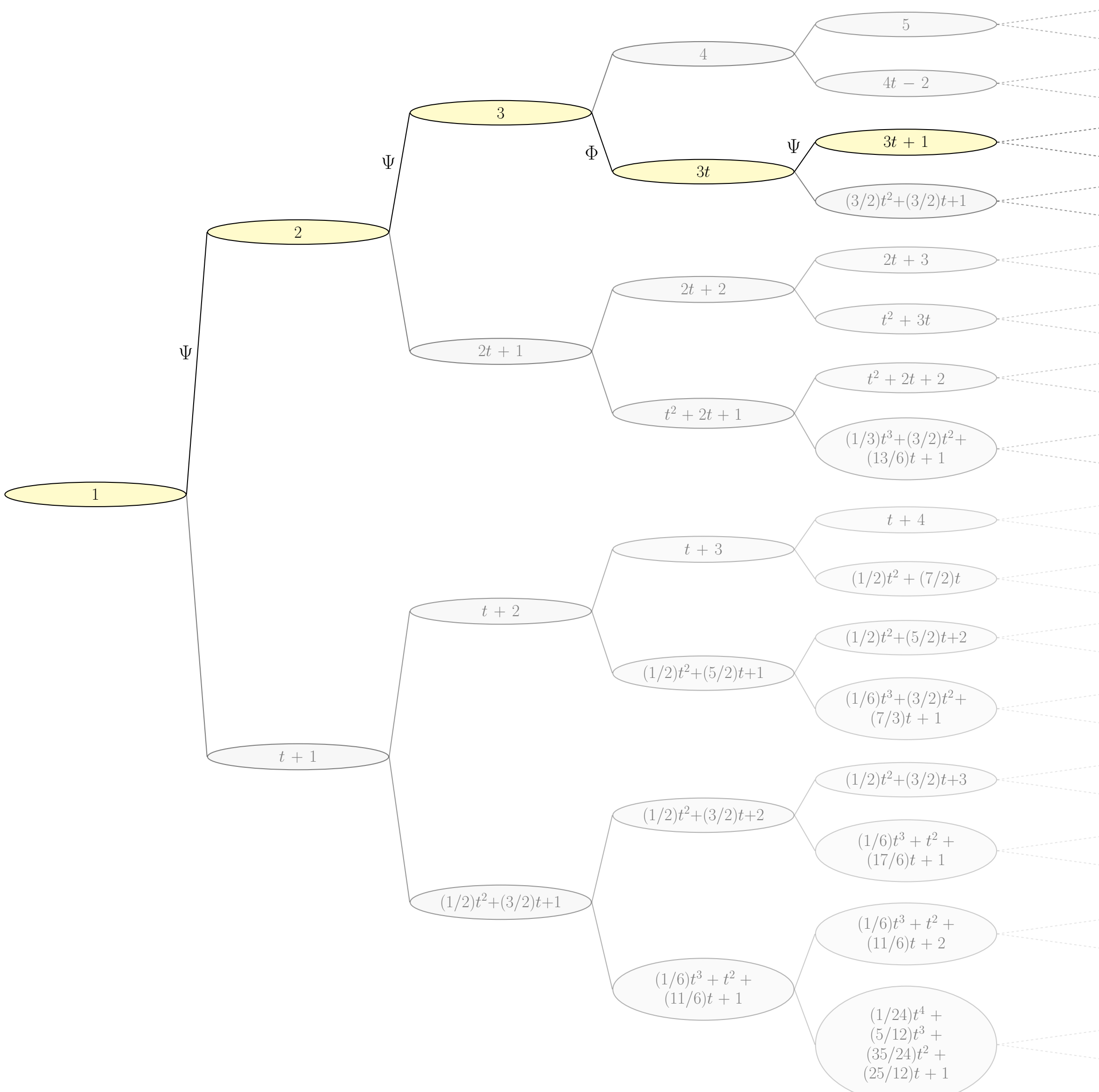


FIGURE 2. The path from $1$ to $\mathsf{p}(t) := 3t + 1$ in the Macaulay tree

## 3. The Forest of Lexicographic Ideals

This section connects lexicographic ideals with the Macaulay tree $\mathcal{M}$. Specifically, Theorem 3.9 shows that $\mathcal{M}$ reappears infinitely many times in the set of saturated lexicographic ideals, with exactly one tree $\mathcal{L}_c$ for each positive codimension $c \in \mathbb{Z}$. To prove this, we study two mappings on the set of lexicographic ideals, defined in analogy with $\Phi$ and $\Psi$. Explicit monomial generators of lexicographic ideals given in terms of Macaulay–Hartshorne expressions help to understand Hilbert polynomials of images of lexicographic ideals under our two mappings.

Lexicographic, or lex-segment, ideals are monomial ideals whose homogeneous pieces are spanned by maximal monomials in lexicographic order. These ideals are central to the classification in [Mac27] of admissible Hilbert polynomials. Their combinatorial nature captures geometric information about Hilbert schemes, as shown by [Har66, PS05, Ree95, RS97] in studying connectedness, radii, and smoothness.

For any vector $u := (u_0, u_1, \dots, u_n) \in \mathbb{N}^{n+1}$, let $x^u := x_0^{u_0} x_1^{u_1} \cdots x_n^{u_n}$. The ***lexicographic ordering*** is the relation $>_{\text{lex}}$ on the monomials in $\mathbb{K}[x_0, x_1, \dots, x_n]$ defined by $x^u >_{\text{lex}} x^v$ if the first nonzero coordinate of $u - v \in \mathbb{Z}^{n+1}$ is positive, where $u, v \in \mathbb{N}^{n+1}$.

**Example 3.1.** We have $x_0 >_{\text{lex}} x_1 >_{\text{lex}} \cdots >_{\text{lex}} x_n$ in lexicographic order on $\mathbb{K}[x_0, x_1, \dots, x_n]$. Further, if $n \geq 2$, then $x_0 x_2^2 >_{\text{lex}} x_1^4 >_{\text{lex}} x_1^3$.

For a homogeneous ideal $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$, lexicographic order gives rise to two monomial ideals associated to $I$. First, the ***lexicographic ideal*** for the Hilbert function $\mathsf{h}_I$ in $\mathbb{K}[x_0, x_1, \dots, x_n]$ is the monomial ideal $L_n^{\mathsf{h}_I}$ whose $i$-th graded piece is spanned by the $\mathsf{h}_{\mathbb{K}[x_0,x_1,\dots,x_n]}(i) - \mathsf{h}_I(i) = \dim_{\mathbb{K}} I_i$ largest monomials in $\mathbb{K}[x_0, x_1, \dots, x_n]_i$, for all $i \in \mathbb{Z}$. The equality $\mathsf{h}_I = \mathsf{h}_{L_n^{\mathsf{h}_I}}$ holds by definition, and $L_n^{\mathsf{h}_I}$ is a homogeneous ideal of $\mathbb{K}[x_0, x_1, \dots, x_n]$; see [Mac27, Section II] or [MS05b, Proposition 2.21]. More importantly, the ***(saturated) lexicographic ideal*** $L_n^{\mathsf{p}_I}$ for the Hilbert polynomial $\mathsf{p}_I$ is the monomial ideal

$$\left(L_n^{\mathsf{h}_I} : \langle x_0, x_1, \dots, x_n \rangle^\infty\right) := \bigcup_{j \geq 1} \left\{ f \in \mathbb{K}[x_0, x_1, \dots, x_n] \mid f \langle x_0, x_1, \dots, x_n \rangle^j \subseteq L_n^{\mathsf{h}_I} \right\}.$$

Saturation with respect to the irrelevant ideal $\langle x_0, x_1, \dots, x_n \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ does not affect the Hilbert function in large degrees, so $L_n^{\mathsf{p}_I}$ also has Hilbert polynomial $\mathsf{p}_I$.

**Example 3.2.** If $X \subset \mathbb{P}^2$ is three distinct noncollinear points, then the Hilbert function of $I_X \subset \mathbb{K}[x_0, x_1, x_2]$ has values $\mathsf{h}_X(\mathbb{N}) = (1, 3, 3, 3, 3, \dots)$. The lexicographic ideal in $\mathbb{K}[x_0, x_1, x_2]$ for $\mathsf{h}_X$ equals $L_2^{\mathsf{h}_X} = \langle x_0^2, x_0 x_1, x_0 x_2, x_1^3 \rangle$. Therefore, the saturation of $L_2^{\mathsf{h}_X}$ with respect to the irrelevant ideal $\langle x_0, x_1, x_2 \rangle$ is $L_2^3 = \langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$, whose Hilbert function has values $\mathsf{h}_{L_2^3}(\mathbb{N}) = (1, 2, 3, 3, 3, 3, \dots)$.

Given a finite sequence of nonnegative integers $a_0, a_1, \dots, a_{n-1} \in \mathbb{N}$, consider the monomial ideal $L(a_0, a_1, \dots, a_{n-1})$ in $\mathbb{K}[x_0, x_1, \dots, x_n]$ with monomial generators

$$\langle x_0^{a_{n-1}+1}, x_0^{a_{n-1}} x_1^{a_{n-2}+1}, \dots, x_0^{a_{n-1}} x_1^{a_{n-2}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, x_0^{a_{n-1}} x_1^{a_{n-2}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle;$$

see [RS97, Notation 1.2]. Lemma 3.3(i) appears in [Moo12, Theorem 2.23].

**Lemma 3.3.** *Let* $\mathsf{p}(t) := \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$*, for integers* $e_0 \geq e_1 \geq \cdots \geq e_d > 0$*, and let* $n \in \mathbb{N}$ *satisfy* $n > d = \deg \mathsf{p}$.

(i) *Define $e_i := 0$, for $d+1 \leq i \leq n$, and $a_j := e_j - e_{j+1}$, for all $0 \leq j \leq n-1$. We have*

$$\begin{aligned} L_n^{\mathsf{p}} &= L(a_0, a_1, \dots, a_{n-1}) \\ &= \langle x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a_d+1}, \\ &\quad x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}+1}, \dots, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle. \end{aligned}$$

(ii) *If there is an integer $0 \leq \ell \leq d-1$ such that $a_j = 0$ for all $j \leq \ell$, and $a_{\ell+1} > 0$, then the minimal monomial generators of $L_n^{\mathsf{p}}$ are given by $m_1, m_2, \dots, m_{n-(\ell+1)}$, where*

$$m_i := x_{i-1}, \text{ for all } 1 \leq i \leq n-(d+1), \text{ and}$$
$$m_{n-d+k} := \left( \prod_{j=0}^{k-1} x_{n-(d+1)+j}^{a_{d-j}} \right) x_{n-(d+1)+k}^{a_{d-k}+1}, \text{ for all } 0 \leq k \leq d-(\ell+1).$$

*If $a_0 \neq 0$, then the minimal monomial generators are those listed in Part (i).*

*Proof.*

(i) Substituting the values $a_j := e_j - e_{j+1}$ determined by the Macaulay–Hartshorne expression of $\mathsf{p}(t)$ in the definition of $L(a_0, a_1, \dots, a_{n-1})$ gives the listed monomials. In degree $a_d + a_{d-1} + \cdots + a_{d-k} + 1$, the monomial $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(d+1)+k}^{a_{d-k}+1}$ is the largest monomial smaller than $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(d+1)+k-1}^{a_{d-(k-1)}+1} x_n^{a_{d-k}}$, for $0 \leq k \leq d-1$. In degree $a_d + a_{d-1} + \cdots + a_0$, the monomial $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-1}^{a_0}$ is similarly the largest monomial smaller than $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1+1} x_n^{a_0-1}$, thus, $L := L(a_0, a_1, \dots, a_{n-1})$ is a lexicographic ideal. For any monomial $g$ in the saturation $\big(L : \langle x_0, x_1, \dots, x_n \rangle^\infty\big)$, there exists $j \in \mathbb{N}$ such that $g x_n^j \in L$. Because the generators of $L$ are not divisible by $x_n$, this implies that $g \in L$, showing that $L$ is saturated.

Before showing that $L$ has the correct Hilbert polynomial, we first prove that the auxiliary ideal $L' := L(0, 0, \dots, 0, a_d, 0, 0, \dots, 0) = \langle x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a_d} \rangle$ has Hilbert polynomial $\sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-a_d}{i+1}$. Setting $S := \mathbb{K}[x_{n-(d+1)}, x_{n-d}, \dots, x_n]$, multiplication by $x_{n-(d+1)}^{a_d}$ defines the first homomorphism in a short exact sequence $0 \to S(-a_d) \to S \to S/\langle x_{n-(d+1)}^{a_d} \rangle \to 0$. Additivity of Hilbert polynomials on short exact sequences shows that $\mathsf{p}_{L'}(t) = \binom{t+d+1}{d+1} - \binom{t+d+1-a_d}{d+1}$. Applying the summation formula [GKP94, p. 159] establishes that $\mathsf{p}_{L'}(t) = \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-a_d}{i+1}$.

We prove the general case by induction on $d := \deg \mathsf{p}$. Suppose that $d = 0$. The ideal $L$ becomes $L := \langle x_0, x_1, \dots, x_{n-2}, x_{n-1}^{a_0} \rangle$, which has constant Hilbert polynomial equal to $a_0 = \mathsf{p}$. Suppose that $d > 0$, let $L' := L(0, 0, \dots, 0, a_d, 0, 0, \dots, 0)$, and let $L'' := L(a_0, a_1, \dots, a_{d-1}, 0, 0, \dots, 0)$. By the short exact sequence

$$0 \to \left(\mathbb{K}[x_0, x_1, \dots, x_n]/L''\right)(-a_d) \to \mathbb{K}[x_0, x_1, \dots, x_n]/L \to \mathbb{K}[x_0, x_1, \dots, x_n]/L' \to 0,$$

where the injection sends $1 \mapsto x_{n-(d+1)}^{a_d}$, we have $\mathsf{p}_L = \mathsf{p}_{L'} + \mathsf{p}_{L''}$. Induction yields $\mathsf{p}_{L''}(t) = \sum_{i=0}^{d-1} \binom{t+i-a_d}{i+1} - \binom{t+i-e_i}{i+1}$, and $\mathsf{p}_{L'}(t) = \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-a_d}{i+1}$ by the previous paragraph, so that $\mathsf{p}_L = \mathsf{p}$. Hence, $L := L(a_0, a_1, \dots, a_{n-1}) = L_n^{\mathsf{p}}$ is the lexicographic ideal for $\mathsf{p}$ in $\mathbb{K}[x_0, x_1, \dots, x_n]$.

(ii) We know that $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} = x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}}$, because either $a_0 = a_1 = \cdots = a_\ell = 0$, or $a_0 \neq 0$ and $\ell = -1$. If $\ell \geq 0$, then the monomial generators

$$x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}+1}, \quad x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+2)}^{a_{\ell+1}} x_{n-(\ell+1)}^{a_\ell+1}, \quad \ldots, \quad x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}$$

from Part (i) are redundant, as they are multiples of the last monomial generator. Removing these redundancies gives the monomial generators $m_1, m_2, \ldots, m_{n-(\ell+1)}$. For all $i \in \{2, 3, \ldots, n-(\ell+1)\}$ and all $j \in \{1, 2, \ldots, i-1\}$ there exists $x_k$ dividing $m_j$ to higher order than the order to which it divides $m_i$, and minimality follows. ☐

To show that $L(a_0, a_1, \ldots, a_{n-1})$ is saturated in the proof of Lemma 3.3, we could alternatively apply Lemma 5.2. The nonminimal list of generators in Lemma 3.3(i) is useful for describing operations on lexicographic ideals in terms of the Macaulay–Hartshorne and Gotzmann partitions of their Hilbert polynomials; see Proposition 3.7. Importantly, Lemma 3.3 shows that all sequences $a_0, a_1, \ldots, a_{n-1}$ of nonnegative integers determine a lexicographic ideal.

The next example uses Lemma 3.3 to identify minimal monomial generators.

**Example 3.4.** The twisted cubic $X \subset \mathbb{P}^3$ has $\mathsf{p}_X(t) = \left[\binom{t+0}{0+1} - \binom{t+0-4}{0+1}\right] + \left[\binom{t+1}{1+1} - \binom{t+1-3}{1+1}\right]$, with lexicographic ideal $L_3^{3t+1} \subset \mathbb{K}[x_0, x_1, x_2, x_3]$; see Example 2.5. We have $d = 1$, $e_0 = 4$, $e_1 = 3$, $e_2 = 0$, and $e_3 = 0$, so that $a_0 = 1$, $a_1 = 3$, and $a_2 = 0$. Hence, applying Lemma 3.3 yields $L_3^{3t+1} = L(1, 3, 0) = \langle x_0, x_1^4, x_1^3 x_2 \rangle$. The Gotzmann number of $3t+1$ is $4$, and can be realized as the sum $a_0 + a_1 + a_2 = 1 + 3 + 0$, as the number $e_0 = 4$, or as the degree of $x_1^3 x_2$.

In analogy with the mapping $\Psi$ , we define the ***lex-expansion*** of any lexicographic ideal $L_n^{\mathsf{p}} := L(a_0, a_1, \ldots, a_{n-1})$ to be the lexicographic ideal $\Psi\big(L_n^{\mathsf{p}}\big) := L(a_0 + 1, a_1, a_2, \ldots, a_{n-1})$. The following lemma explains our choice of notation for lex-expansion.

**Lemma 3.5.** *Let $\mathsf{p}$ be an admissible Hilbert polynomial and $n > \deg \mathsf{p}$ a positive integer. We have $\Psi\big(L_n^{\mathsf{p}}\big) = L_n^{\Psi(\mathsf{p})}$, and the mapping $\Psi$ on lexicographic ideals preserves codimension.*

*Proof.* See Lemma 5.5. Note that $n - \deg \Psi(\mathsf{p}) = n - \deg \mathsf{p}$. ☐

**Example 3.6.** The Hilbert polynomial of a planar cubic curve $C \subset \mathbb{P}^3$ is $3t$, with Macaulay–Hartshorne expression $\left[\binom{t+0}{0+1} - \binom{t+0-3}{0+1}\right] + \left[\binom{t+1}{1+1} - \binom{t+1-3}{1+1}\right]$. Applying Lemma 3.3, we have $a_0 = 0$, $a_1 = 3$, and $a_2 = 0$, so $L_3^{3t} = L(0, 3, 0) = \langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$. Lemma 3.5 shows that $\Psi\big(L(0, 3, 0)\big) = L(1, 3, 0) = \langle x_0, x_1^4, x_1^3 x_2 \rangle$; compare Example 3.4.

In analogy with $\Phi$, for any ideal $I \subseteq \mathbb{K}[x_0, x_1, \ldots, x_n]$, we denote the ***extension*** ideal by $\Phi(I) := I \cdot \mathbb{K}[x_0, x_1, \ldots, x_{n+1}]$. For lexicographic ideals, we have an analogue of Lemma 3.5.

**Proposition 3.7.** *Let $\mathsf{p}$ be an admissible Hilbert polynomial and $n > \deg \mathsf{p}$ a positive integer. We have $\Phi\big(L_n^{\mathsf{p}}\big) = L_{n+1}^{\Phi(\mathsf{p})}$ and $\Phi\big(L(a_0, a_1, \ldots, a_{n-1})\big) = L(0, a_0, a_1, \ldots, a_{n-1})$ holds for all $a_0, a_1, \ldots, a_{n-1} \in \mathbb{N}$, so extension preserves codimension.*

*Proof.* Let $\mathsf{q}$ denote the Hilbert polynomial of $\Phi\big(L_n^{\mathsf{p}}\big) \subset \mathbb{K}[x_0, x_1, \ldots, x_{n+1}]$. The extension $\Phi\big(L_n^{\mathsf{p}}\big)$ is generated by the images of the generators of $L_n^{\mathsf{p}}$ under the inclusion mapping $\mathbb{K}[x_0, x_1, \ldots, x_n] \hookrightarrow \mathbb{K}[x_0, x_1, \ldots, x_{n+1}]$, namely

$$\Phi\big(L_n^{\mathsf{p}}\big) = \langle x_0, x_1, \ldots, x_{n-(d+2)}, x_{n-(d+1)}^{a_d+1}, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}+1}, \ldots, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, \\ x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle \subset \mathbb{K}[x_0, x_1, \ldots, x_{n+1}].$$

Because $\mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ has $n+2$ variables, we reindex the powers in this list of generators to describe $\mathsf{q}$. For all $i \in \{1, 2, \dots, d+1\}$, set $a'_i := a_{i-1}$, and set $a'_0 := 0$. Rewriting the monomial generators and adding $x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d} \cdots x_{n-2}^{a'_2} x_{n-1}^{a'_1+1}$ to the list, we find that

$$\Phi\big(L_n^{\mathsf{p}}\big) = \langle x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a'_{d+1}+1}, x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d+1}, \dots, x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d} \cdots x_{n-2}^{a'_2} x_{n-1}^{a'_1+1},$$
$$x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d} \cdots x_{n-1}^{a'_1} x_n^{a'_0} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}].$$

This gives monomial generators of $\Phi(L_n^{\mathsf{p}})$ in the form of Lemma 3.3(i). By Lemma 2.4, the first $a'_{d+1} = a_d$ parts in the Gotzmann partition of $\mathsf{q}$ equal $d+1$, the next $a'_d = a_{d-1}$ parts in the Gotzmann partition equal $d$, and so on. Hence, every part in the Gotzmann partition of $\mathsf{p}$ has increased by $1$, and $\mathsf{q} = \Phi(\mathsf{p})$. Finally, the codimension of $\Phi\big(L_n^{\mathsf{p}}\big) = L_{n+1}^{\Phi(\mathsf{p})}$ equals $n+1-\deg \Phi(\mathsf{p}) = n - \deg \mathsf{p}$, thus $\Phi$ preserves codimension. ☐

**Example 3.8.** Example 3.4 and Proposition 3.7 show that $\Phi\big(L_3^{3t+1}\big) = \langle x_0, x_1^4, x_1^3 x_2\rangle$ in $\mathbb{K}[x_0, x_1, x_2, x_3, x_4]$ is the lexicographic ideal with Hilbert polynomial $\Phi(3t+1) = \frac{3}{2}t^2 + \frac{5}{2}t - 1$.

The following describes a forest structure on the set of lexicographic ideals.

**Theorem 3.9.** *For each positive codimension $c \in \mathbb{Z}$, the graph $\mathcal{L}_c$ whose vertex set consists of all lexicographic ideals of codimension $c$ and whose edges are all possible pairs of the form $\Big(L_n^{\mathsf{p}}, L_n^{\Psi(\mathsf{p})}\Big)$ and $\Big(L_n^{\mathsf{p}}, L_n^{\Phi(\mathsf{p})}\Big)$, where $L_n^{\mathsf{p}}$ is a lexicographic ideal of codimension $c$, is an infinite binary tree. The root of the binary tree $\mathcal{L}_c$ is the lexicographic ideal $L_c^1 \subset \mathbb{K}[x_0, x_1, \dots, x_c]$.*

*Proof.* Lemma 3.5 and Proposition 3.7 combined with Lemma 3.3 show that the mapping $\mathcal{M} \to \mathcal{L}_c$ defined by $\mathsf{p} \mapsto L_{c+\deg(\mathsf{p})}^{\mathsf{p}}$ is a graph isomorphism. The root of $\mathcal{L}_c$ is the ideal $L_c^1 := \langle x_0, x_1, \dots, x_{c-1}\rangle$ in $\mathbb{K}[x_0, x_1, \dots, x_c]$, which has Hilbert polynomial $1$. ☐

For positive $c \in \mathbb{Z}$, we call the tree $\mathcal{L}_c$ of Theorem 3.9 the ***lexicographic tree of codimension*** $c$, and we call the union $\mathcal{L} := \bigsqcup_{c \in \mathbb{N}, c>0} \mathcal{L}_c$ the ***lexicographic forest***. The vertices of $\mathcal{L}_c$ are ideals living in infinitely many different polynomial rings.

## 4. Random Schemes in the Hilbert Forest

In this section, we transform the set of nonempty Hilbert schemes into a discrete probability space, exploiting the graph structure on the set of lexicographic ideals. This allows probabilistic statements about the geometry of random Hilbert schemes. In Examples 4.5–4.7, we regard the dimensions and degrees of subschemes parametrized by Hilbert schemes, and the radii of Hilbert schemes, as random variables and we estimate their expected values and variances.

**Theorem 4.1.** *For each positive codimension $c \in \mathbb{Z}$, the graph $\mathcal{H}_c$ whose vertex set consists of every nonempty Hilbert scheme $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ that parametrizes codimension $c$ subschemes of some projective space $\mathbb{P}^n$ and whose edges are all pairs of the form $\Big(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n), \mathrm{Hilb}^{\Psi(\mathsf{p})}(\mathbb{P}^n)\Big)$ and $\Big(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n), \mathrm{Hilb}^{\Phi(\mathsf{p})}(\mathbb{P}^{n+1})\Big)$, where $\mathsf{p}$ is an admissible Hilbert polynomial and $n := c + \deg \mathsf{p}$, is an infinite binary tree. The root of the tree $\mathcal{H}_c$ is the Hilbert scheme $\mathrm{Hilb}^1(\mathbb{P}^c)$.*

For positive $c \in \mathbb{Z}$, we call the tree $\mathcal{H}_c$ of Theorem 4.1 the ***Hilbert tree of codimension*** $c$, and we call the disjoint union $\mathcal{H} := \bigsqcup_{c \in \mathbb{N}, c>0} \mathcal{H}_c$ the ***Hilbert forest***.

*Proof.* Each pair of an admissible Hilbert polynomial $\mathsf{p}$ and positive $c \in \mathbb{Z}$ uniquely determines both a lexicographic ideal $L_n^{\mathsf{p}}$ and a Hilbert scheme $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$, where $n := c + \deg \mathsf{p}$. The specified graph structure makes this correspondence into a graph isomorphism. $\square$

Hilbert schemes in a fixed $\mathcal{H}_c$ parametrize subschemes in infinitely many different projective spaces.

**Example 4.2.** The ray $(\mathrm{Hilb}^1(\mathbb{P}^c), \mathrm{Hilb}^2(\mathbb{P}^c), \dots, \mathrm{Hilb}^{\Psi^k(1)}(\mathbb{P}^c), \dots)$ of $\mathcal{H}_c$ contains all Hilbert schemes of points in $\mathbb{P}^c$. The ray $(\mathrm{Hilb}^1(\mathbb{P}^c), \mathrm{Hilb}^{t+1}(\mathbb{P}^{c+1}), \dots, \mathrm{Hilb}^{\Phi^{n-c}(1)}(\mathbb{P}^n), \dots)$ of $\mathcal{H}_c$ contains all Grassmannians parametrizing linear subspaces of a fixed codimension $c$, as $\mathbb{G}(n-c, n) = \mathrm{Hilb}^{\Phi^{n-c}(1)}(\mathbb{P}^n)$ .

**Remark 4.3.** The edges of the graphs $\mathcal{M}$ and $\mathcal{L}$ arise from combinatorial and algebraic operations on their vertices. The edges of the Hilbert forest arise from rational mappings.

Consider an edge $\left(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n), \mathrm{Hilb}^{\Psi(\mathsf{p})}(\mathbb{P}^n)\right)$ in $\mathcal{H}$. For any $x \in \mathbb{P}^n$, there is a rational mapping $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n) \dashrightarrow \mathrm{Hilb}^{\Psi(\mathsf{p})}(\mathbb{P}^n)$ with $[X] \mapsto [X \cup \{x\}]$ for all $X$ such that $x \notin X$. This does not extend to a regular mapping, even in elementary cases. For instance, to complete such a mapping $\varphi\colon \mathrm{Hilb}^1(\mathbb{P}^2) \dashrightarrow \mathrm{Hilb}^2(\mathbb{P}^2)$, the value $\varphi(x)$ must parametrize a double point at $x$, which is a subscheme of every line of approach through $x$. Hence, $\varphi$ has no well-defined value at $x$.

The ***lexicographic point*** of $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ is the point $[L_n^{\mathsf{p}}]$. It is nonsingular and lies on a unique irreducible component $(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n))^{lex} \subseteq \mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ called the ***lexicographic component***; see [RS97]. Consider an edge of the form $\left(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n), \mathrm{Hilb}^{\Phi(\mathsf{p})}(\mathbb{P}^{n+1})\right)$ in $\mathcal{H}$. If $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ defines $X_I \subset \mathbb{P}^n$, then $\Phi(I)$ defines the join $\mathrm{Join}(X_I, x)$ of $X_I$ and $x := [0 : 0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$. Thus, Proposition 3.7 shows that $\mathrm{Join}\left(X_{L_n^{\mathsf{p}}}, x\right) = X_{\Phi\left(L_n^{\mathsf{p}}\right)}$ has Hilbert polynomial $\Phi(\mathsf{p})$. By change of coordinates, this is true for general points on $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n))^{lex}$, thus there is a rational mapping $(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n))^{lex} \dashrightarrow \mathrm{Hilb}^{\Phi(\mathsf{p})}(\mathbb{P}^{n+1})$ defined by $X_I \mapsto X_{\Phi(I)}$. This mapping cannot be extended to other irreducible components of $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$, or even to the intersection of the lexicographic component with other irreducible components, due to the presence of nonlexicographic Borel points on these components; see [Ree95, Remark 2.2] and Theorem 7.4.

**Remark 4.4.** For fixed positive $n \in \mathbb{Z}$, nonempty Hilbert schemes $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ are found in the Hilbert trees $\mathcal{H}_c$, for all $1 \leq c \leq n$. Given $1 \leq c \leq n$, these are all vertices in $\mathcal{H}_c$ with corresponding admissible Hilbert polynomials of degree $n - c$.

Consider the probability measure $\mathrm{Pr}\colon 2^{\mathcal{H}_c} \to [0,1]$ with sample space $\mathcal{H}_c$ determined by a normalized nonnegative function $\mathrm{pr}\colon \mathcal{H}_c \to \mathbb{R}$; see [Bil95, Examples 2.8–2.9]. To mimic uniform distribution, every vertex of $\mathcal{H}_c$ at a fixed height is equally likely; given a mass function $f_c\colon \mathbb{N} \to [0,1]$, let $\mathrm{pr}\big(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)\big) := f_c(k)/2^k$, for all $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n) \in \mathcal{H}_c$ at height $k$. Elementary choices include the ***geometric*** mass function $f_c(k) := p_c(1-p_c)^k$, for $0 < p_c < 1$, and the ***Poisson*** mass function $f_c(k) := e^{-\lambda_c}\lambda_c^k/k!$, for $\lambda_c > 0$. If $f_c$ is geometric, then $p_c = f_c(0)/2^0$ is the likelihood of the root of $\mathcal{H}_c$. For both distributions, every vertex has nonzero probability. Distributions on $\mathcal{H}$ are specified via a function $f_c\colon \mathbb{N} \to [0,1]$ for each $\mathcal{H}_c$ and a mass function $f\colon \mathbb{N}\setminus\{0\} \to [0,1]$, by setting $\mathrm{pr}\left(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)\right) := f(c)f_c(k)/2^k$, for all $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n) \in \mathcal{H}_c$ at height $k$. If all $f_c$ are equal, then expected values and variances of random variables on $\mathcal{H}$ can be computed by restricting to any $\mathcal{H}_c$.

**Example 4.5.** For positive $c \in \mathbb{Z}$, let $\operatorname{pdm}_c\colon \mathcal{H}_c \to \mathbb{N}$ be the ***parametrized dimension***, defined by $\operatorname{pdm}_c\big(\operatorname{Hilb}^{\mathsf{p}}(\mathbb{P}^n)\big) := \deg \mathsf{p}$. This random variable captures dimensions of subschemes parametrized by random Hilbert schemes. Let $f_c\colon \mathbb{N} \to [0,1]$ be a mass function. We have $\Pr\big(\operatorname{pdm}_c = d\big) = \sum_{k\in\mathbb{N}} \binom{k}{d} f_c(k)/2^k$, as vertices have height $k$ and $\operatorname{pdm}_c = d$ with probability $\binom{k}{d} f_c(k)/2^k$. Hence, the expected value is $\sum_{d\in\mathbb{N}} d \sum_{k\in\mathbb{N}} \binom{k}{d} f_c(k)/2^k$. Applying the identity $\sum_{d=0}^{k} d\binom{k}{d} = 2^{k-1}k$, for all positive $K \in \mathbb{Z}$ we have

$$\sum_{d=0}^{K} d \sum_{k=0}^{K} \binom{k}{d} \frac{f_c(k)}{2^k} = \sum_{k=1}^{K}\sum_{d=1}^{k} d\binom{k}{d}\frac{f_c(k)}{2^k} = \sum_{k=1}^{K} 2^{k-1}k\frac{f_c(k)}{2^k} = \frac{1}{2}\sum_{k=1}^{K} k f_c(k) \to \frac{1}{2}\operatorname{E}\big(f_c\big).$$

If $f_c$ is geometric, then we obtain $\operatorname{E}\big(\operatorname{pdm}_c\big) = (1-p_c)/2p_c$, where $p_c = f_c(0)$. If $f_c$ is Poisson with mean $\lambda_c > 0$, then we have $\operatorname{E}\big(\operatorname{pdm}_c\big) = \lambda_c/2$.

We similarly compute $\operatorname{E}\big(\operatorname{pdm}_c^2\big)$ by applying the identity $\sum_{d=0}^{k} d\binom{k}{d} = 2^{k-1}k$, obtaining

$$\sum_{k=1}^{K} k\frac{f_c(k)}{2^k}\sum_{d=0}^{k-1}(d+1)\binom{k-1}{d} = \sum_{k=1}^{K} k\frac{f_c(k)}{2^k}2^{k-2}(k+1) = \sum_{k=1}^{K} k(k+1)\frac{f_c(k)}{4},$$

which converges to $\frac{1}{4}\left(\operatorname{E}\big(f_c^2\big) + \operatorname{E}\big(f_c\big)\right)$. We then have

$$\operatorname{var}\big(\operatorname{pdm}_c\big) = \frac{1}{4}\left(\operatorname{E}\big(f_c^2\big) + \operatorname{E}\big(f_c\big)\right) - \left(\frac{\operatorname{E}\big(f_c\big)}{2}\right)^2 = \frac{1}{4}\left(\operatorname{var}\big(f_c\big) + \operatorname{E}\big(f_c\big)\right).$$

If $f_c$ is geometric, then we have $\operatorname{var}\big(\operatorname{pdm}_c\big) = (1-p_c^2)/4p_c^2$, while if $f_c$ is Poisson, then we have $\operatorname{var}\big(\operatorname{pdm}_c\big) = \lambda_c/2$.

**Example 4.6.** Let $\operatorname{rad}_c\colon \mathcal{H}_c \to \mathbb{N}$ map any Hilbert scheme $\operatorname{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ to its ***radius***, defined as the radius of the incidence graph of components of $\operatorname{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$. The inequality $\operatorname{rad}_c \leq 1 + \operatorname{pdm}_c$ holds by [Ree95, Theorem 7], so for all $\operatorname{Hilb}^{\mathsf{p}}(\mathbb{P}^n) \in \mathcal{H}_c$ and $r \in \mathbb{N}$, if we have $\operatorname{pdm}_c\big(\operatorname{Hilb}^{\mathsf{p}}(\mathbb{P}^n)\big) \leq r-1$, then we have $\operatorname{rad}_c\big(\operatorname{Hilb}^{\mathsf{p}}(\mathbb{P}^n)\big) \leq r$, which implies that $\Pr\big(\operatorname{pdm}_c \leq r-1\big) \leq \Pr\big(\operatorname{rad}_c \leq r\big)$ holds. The likelihood that every component intersects the lexicographic component satisfies

$$\Pr\big(\operatorname{rad}_c \leq 1\big) \geq \Pr\big(\operatorname{pdm}_c \leq 0\big) = \Pr\big(\operatorname{pdm}_c = 0\big) = \sum_{k\in\mathbb{N}} \frac{f_c(k)}{2^k}.$$

If $f_c$ is geometric, then this yields $\Pr\big(\operatorname{rad}_c \leq 1\big) \geq 2p_c/(1+p_c)$; for instance, $p_c := 1/2$ implies $\Pr\big(\operatorname{rad}_c \leq 1\big) \geq 2/3$. If $f_c(k) := e^{-\lambda_c}\lambda_c^k/k!$ is Poisson, then the series equals $e^{-\lambda_c/2}$. Similarly, if $f_c$ is geometric, then $\Pr\big(\operatorname{rad}_c \leq 2\big) \geq 4p_c/(1+p_c)^2$ holds, so that $\Pr\big(\operatorname{rad}_c \leq 2\big) \geq 8/9$ for $p_c := 1/2$. If $f_c$ is Poisson, then $\Pr\big(\operatorname{rad}_c \leq 2\big) \geq (1+\lambda_c/2)e^{-\lambda_c/2}$. Hence, depending on the underlying distribution, random Hilbert schemes can have small radii with high probability.

The expected value is independent of the underlying distribution in the next example.

**Example 4.7.** The ***parametrized degree*** $\operatorname{pdg}_c\colon \mathcal{H}_c \to \mathbb{N}$ is as follows: let $\operatorname{LC}_c\colon \mathcal{H}_c \to \mathbb{Q}$ map $\operatorname{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ to the leading coefficient of $\mathsf{p}(t) \in \mathbb{Q}[t]$ and set $\operatorname{pdg}_c := (\operatorname{pdm}_c!)(\operatorname{LC}_c)$. This random variable maps a Hilbert scheme to the degree of subschemes it parametrizes. Vertices with $\operatorname{pdg}_c = d$ are $\operatorname{Hilb}^d(\mathbb{P}^c)$ and the subtree rooted at $\operatorname{Hilb}^{\Phi(d)}(\mathbb{P}^{c+1})$. For all $k \in \mathbb{N}$ and $1 \leq d \leq k$, there are $2^{k-d}$ vertices with $\operatorname{pdg}_c = d$ and one with $\operatorname{pdg}_c = k+1$. Thus, we have $\Pr\big(\operatorname{pdg}_c = d\big) = \left(\frac{f_c(d-1)}{2^{d-1}} + \sum_{k\geq d}\frac{2^{k-d}f_c(k)}{2^k}\right) = \frac{1}{2^{d-1}}\left(f_c(d-1) + \sum_{k\geq d}\frac{f_c(k)}{2}\right) < \frac{1}{2^{d-1}}$.

This gives $\mathrm{E}\big(\mathrm{pdg}_c^2\big) = \sum_{d>0} d^2 \Pr\big(\mathrm{pdg}_c = d\big) \leq \sum_{d>0} \frac{d^2}{2^{d-1}} = 12$. Hence, the inequality $0 \leq \mathrm{var}\big(\mathrm{pdg}_c\big) = \mathrm{E}\big(\mathrm{pdg}_c^2\big) - \mathrm{E}\big(\mathrm{pdg}_c\big)^2$ yields $\mathrm{E}\big(\mathrm{pdg}_c\big) \leq \sqrt{12} \approx 3.46$.

## 5. Borel Ideals

This section examines analogues of $\Phi$ and $\Psi$ for Borel ideals, concluding with a well-known algorithm that generates saturated Borel ideals.

A monomial ideal $I \subseteq \mathbb{K}[x_0, x_1, \ldots, x_n]$ is ***Borel***, or ***strongly stable***, if, for all monomials $m \in I$, for all $x_j$ dividing $m$, and for all $x_i >_{\mathrm{lex}} x_j$, we have $m x_i x_j^{-1} \in I$. In characteristic $0$, this is equivalent to being fixed by the linear action of upper triangular matrices in $\mathrm{GL}_{n+1}(\mathbb{K})$; see [BS87a, Proposition 2.7]. For any monomial $m$, let $\max m$ be the maximum integer $j$ such that the variable $x_j$ divides $m$, and $\min m$ be the minimum such integer.

**Example 5.1.** The monomial ideal $I = \langle x_0^2, x_0 x_1, x_1^2 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$ is Borel. The monomial $m := x_1^5 x_2 x_7^2 \in \mathbb{K}[x_0, x_1, \ldots, x_{13}]$ satisfies $\max m = 7$ and $\min m = 1$.

We collect some useful properties of Borel ideals.

**Lemma 5.2.** *Let $I \subseteq \mathbb{K}[x_0, x_1, \ldots, x_n]$ be a monomial ideal.*

- *(i) The ideal $I$ is Borel if and only if, for all minimal monomial generators $g \in I$, for all variables $x_j$ dividing $g$, and for all $x_i >_{\mathrm{lex}} x_j$, we have $g x_i x_j^{-1} \in I$.*
- *(ii) If, for all minimal monomial generators $g' \in I$, for all variables $x_j$ dividing $g'$, and for all $x_i >_{\mathrm{lex}} x_j$, we have $g' x_i x_j^{-1} \in I$, then, for all monomials $m \in I$, there exists a unique minimal monomial generator $g \in I$ and unique monomial $m' \in \mathbb{K}[x_0, x_1, \ldots, x_n]$ such that $m = g m'$ and $\max g \leq \min m'$.*
- *(iii) If $I$ is a Borel ideal, then $I$ is saturated with respect to the irrelevant ideal $\langle x_0, x_1, \ldots, x_n \rangle$ if and only if the minimal monomial generators of $I$ are not divisible by the variable $x_n$.*
- *(iv) If $I$ is Borel with constant Hilbert polynomial $\mathsf{p}_I \in \mathbb{N}$, then there exists an integer $k \in \mathbb{N}$ such that $x_{n-1}^k \in I$.*

*Proof.*

- (i) If $I$ is Borel, then this holds for all $g \in I$. Conversely, let $m \in I$ be any monomial. By (ii), there is a unique factorization $m = g m'$, where $g \in I$ is a minimal generator and $m' \in \mathbb{K}[x_0, x_1, \ldots, x_n]$ is a monomial such that $\max g \leq \min m'$. Let $x_j$ divide $m$ and let $x_i >_{\mathrm{lex}} x_j$. Either $x_j$ divides $g$, in which case $g x_i x_j^{-1} \in I$ and so $m x_i x_j^{-1} \in I$, or $x_j$ divides $m'$, in which case $m x_i x_j^{-1}$ is a multiple of $g$.
- (ii) See the proof of [MS05b, Lemma 2.11].
- (iii) If $x_n$ divides a minimal generator $g \in I$, then for all $x_j$ we have $(g x_n^{-1}) x_j \in I$, while $g x_n^{-1} \notin I$. Conversely, any monomial $m \in (I : \langle x_0, x_1, \ldots, x_n \rangle) \setminus I$ yields a minimal monomial generator $m x_n \in I$, for example by Part (ii).
- (iv) See the proof of [Moo12, Lemma 3.17]. □

**Example 5.3.** Lemmas 3.3 and 5.2(i) show that lexicographic ideals are Borel. If $x_j \mid m_k$ holds, for $L_n^{\mathsf{p}} = \langle m_1, m_2, \ldots, m_{n-(\ell+1)} \rangle$, then $m_k x_i x_j^{-1} \in \langle m_1, m_2, \ldots, m_{k-1} \rangle$ holds, for $i < j$.

We generate saturated Borel ideals via expansions. If $I \subseteq \mathbb{K}[x_0, x_1, \ldots, x_n]$ is a saturated Borel ideal, then a minimal monomial generator $g \in I$ is ***expandable*** if the set $\big\{ g x_{i+1} x_i^{-1} \mid x_i \text{ divides } g \text{ and } 0 \leq i < n-1 \big\}$ contains no minimal monomial generators of $I$. The ***expansion*** of $I$ at an expandable generator $g$ is the monomial ideal

$$I' := \langle I \setminus \langle g \rangle \rangle + \langle g x_j \mid \max g \leq j \leq n-1 \rangle \subset \mathbb{K}[x_0, x_1, \ldots, x_n];$$

see [Moo12, Definition 3.4]. The monomial $1 \in \langle 1 \rangle$ is vacuously expandable with expansion $\langle x_0, x_1, \ldots, x_{n-1} \rangle \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$. Lemma 5.2(i),(iii) ensure that the expansion of a saturated Borel ideal is again saturated and Borel.

**Example 5.4.** Let $L_n^{\mathsf{p}} = L(a_0, a_1, \ldots, a_{n-1}) = \langle m_1, m_2, \ldots, m_{n-(\ell+1)} \rangle$ be lexicographic, as in Lemma 3.3. The last generator is $m_{n-(\ell+1)} := x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}}$. If $a_i > 0$, then $x_{n-(i+1)}$ divides $m_{n-(\ell+1)} x_{n-i} x_{n-(i+1)}^{-1}$ to order $a_i - 1$, which is not the case for any $m_j$. Therefore, the expansion at $m_{n-(\ell+1)}$ has generators

$$\Big\{ m_1, m_2, \ldots, m_{n-(\ell+2)}, m_{n-(\ell+1)} x_{n-(\ell+2)}, m_{n-(\ell+1)} x_{n-(\ell+1)}, \ldots, m_{n-(\ell+1)} x_{n-1} \Big\}$$

$$= \Big\{ x_0,\ x_1,\ \ldots,\ x_{n-(d+2)},$$
$$x_{n-(d+1)}^{a_d+1},\ x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}+1},\ \ldots,\ x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+4)}^{a_{\ell+3}} x_{n-(\ell+3)}^{a_{\ell+2}+1},$$
$$x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}+1},\ x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}} x_{n-(\ell+1)},\ \ldots$$
$$\ldots,\ x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}} x_{n-1} \Big\}.$$

These are exactly the minimal monomial generators of the ideal $L(a_0 + 1, a_1, a_2, \ldots, a_{n-1})$. Hence, the expansion of $L_n^{\mathsf{p}}$ at $m_{n-(\ell+1)}$ equals $L(a_0 + 1, a_1, a_2, \ldots, a_{n-1}) = \Psi\big(L_n^{\mathsf{p}}\big)$.

Lemma 5.5 describes Hilbert polynomials of expansion ideals.

**Lemma 5.5.** *If $I'$ is any expansion of a saturated Borel ideal $I$, then we have $\mathsf{p}_{I'} = \Psi(\mathsf{p}_I)$.*

*Proof.* Let $I'$ be the expansion of $I$ at $g$. For all $d \geq \deg g$, Lemma 5.2(ii) shows that the only monomial in $I_d \setminus I'_d$ is $g x_n^{d - \deg g}$, so that $\mathsf{h}_{I'}(d) = 1 + \mathsf{h}_I(d)$. Hence, we have $\mathsf{p}_{I'} = 1 + \mathsf{p}_I$. $\square$

**Example 5.6.** The ideal $I := \langle x_0^2, x_0 x_1, x_1^2 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$ is not lexicographic, but is saturated and Borel by Lemma 5.2(i),(iii), and we have $\mathsf{p}_I(t) = 3$. Because both $x_0^2 \cdot x_1 x_0^{-1} = x_0 x_1$ and $x_0 x_1 \cdot x_1 x_0^{-1} = x_1^2$ are minimal monomial generators of $I$, the only expandable generator of $I$ is $x_1^2$, with expansion $I' := \langle x_0^2, x_0 x_1, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$ such that $\mathsf{p}_{I'}(t) = 4 = \Psi(3)$.

For a saturated Borel ideal $I \subseteq \mathbb{K}[x_0, x_1, \ldots, x_n]$, let $\nabla(I) \subseteq \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$ be the image of $I$ under the mapping $\mathbb{K}[x_0, x_1, \ldots, x_n] \to \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$ defined by $x_j \mapsto x_j$ for $0 \leq j \leq n-2$ and $x_k \mapsto 1$ for $k \in \{n-1, n\}$. The saturation of $I \cap \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$ with respect to $\langle x_0, x_1, \ldots, x_{n-1} \rangle$, and $\big(\operatorname{sat}_{x_{n-1}, x_n} I\big) \cap \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$, both give $\nabla(I)$, where $\operatorname{sat}_{x_{n-1}, x_n} I$ is the ***double saturation*** of $I$ in $\mathbb{K}[x_0, x_1, \ldots, x_n]$; see [Ree95, p. 642].

**Lemma 5.7.** *If $I$ is a saturated Borel ideal, then we have $\mathsf{p}_{\nabla(I)} = \nabla(\mathsf{p}_I)$.*

*Proof.* Let $S := \mathbb{K}[x_0, x_1, \ldots, x_n]$. Because $I$ is saturated, $x_n$ is a nonzerodivisor, and

$$0 \longrightarrow (S/I)(-1) \longrightarrow S/I \longrightarrow \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]/I \cap \mathbb{K}[x_0, x_1, \ldots, x_{n-1}] \longrightarrow 0$$

is a short exact sequence. Thus, for all $i \in \mathbb{Z}$, the Hilbert function of $I \cap \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$ satisfies $\mathsf{h}_{I \cap \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]}(i) = \mathsf{h}_I(i) - \mathsf{h}_I(i-1)$. Hence, by saturating $I \cap \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$ with respect to $\langle x_0, x_1, \ldots, x_{n-1} \rangle$, we find that $\mathsf{p}_{\nabla(I)}(t) = \mathsf{p}_I(t) - \mathsf{p}_I(t-1) = [\nabla(\mathsf{p}_I)](t)$. $\square$

**Example 5.8.** The saturated Borel ideal $I := \langle x_0^2, x_0 x_1, x_0 x_2, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ has Hilbert polynomial $\mathsf{p}_I(t) = 3t + 1$. We have $\nabla(I) = \langle x_0^2, x_0 x_1, x_0, x_1^3 \rangle = \langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$ with $\mathsf{p}_{\nabla(I)}(t) = 3 = \nabla(3t + 1)$; see [PS85, Lemma 6] for more about $I$.

The following lemma generalizes Proposition 3.7 to arbitrary saturated Borel ideals.

**Lemma 5.9.** *For all saturated Borel ideals $I$, there exists $j \in \mathbb{N}$ such that $\mathsf{p}_{\Phi(I)} = \Psi^j \, \Phi(\mathsf{p}_I)$.*

*Proof.* The ideal $\Phi(I)$ is saturated and Borel by Lemma 5.2(i),(iii), so has no minimal monomial generator divisible by $x_n$. Thus, $\nabla\big(\Phi(I)\big) = I$. Lemma 5.7 implies that $\nabla(\mathsf{p}_{\Phi(I)}) = \mathsf{p}_I$, so that $\Phi \, \nabla(\mathsf{p}_{\Phi(I)}) = \Phi(\mathsf{p}_I)$. To finish, Lemma 2.7(iii) shows that $\mathsf{p}_{\Phi(I)} - \Phi \, \nabla(\mathsf{p}_{\Phi(I)}) \in \mathbb{N}$. $\square$

**Example 5.10.** The extension ideal $\Phi(I) := \langle x_0^2, x_0x_1, x_1^2 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ of the ideal $I$ in Example 5.6 has Hilbert polynomial $3t + 1 = \Psi \, \Phi(3)$.

The following is the heart of the algorithm in [Ree92] generating saturated Borel ideals.

**Theorem 5.11.** *If $I \neq \langle 1 \rangle$ is a saturated Borel ideal of codimension $c$, then there is a finite binary sequence of expansions and extensions from $\langle x_0, x_1, \ldots, x_{c-1} \rangle \subset \mathbb{K}[x_0, x_1, \ldots, x_c]$ to $I$.*

*Proof.* See [Moo12, Theorem 3.20]. $\square$

The binary sequences from Theorem 5.11 generating an ideal are not generally unique, but they are for lexicographic ideals, by Theorem 3.9. Theorem 5.11 leads to the following algorithm; see [Ree92, Appendix A], [Moo12, Chapter 3], and [CLMR11, Section 5].

**Algorithm 5.12.**

Input: an admissible Hilbert polynomial $\mathsf{p} \in \mathbb{Q}[t]$ and $n \in \mathbb{N}$ satisfying $n > \deg \mathsf{p}$
Output: all saturated Borel ideals with Hilbert polynomial $\mathsf{p}$ in $\mathbb{K}[x_0, x_1, \ldots, x_n]$

$j := 0$; $d := \deg \mathsf{p}$;
$\mathsf{q}_0 := \nabla^d(\mathsf{p})$; $\mathsf{q}_1 := \nabla^{d-1}(\mathsf{p})$; $\ldots$; $\mathsf{q}_{d-1} := \nabla^1(\mathsf{p})$; $\mathsf{q}_d := \mathsf{p}$;
$\mathcal{S} := \{\langle 1 \rangle\}$, where $\langle 1 \rangle$ is the unit ideal in $\mathbb{K}[x_0, x_1, \ldots, x_{n-d}]$;
WHILE $j \leq d$ DO
    $\mathcal{T} := \varnothing$;
    FOR $J \in \mathcal{S}$, considered as an ideal in $\mathbb{K}[x_0, x_1, \ldots, x_{n-d+j}]$ DO
        IF $\mathsf{q}_j - \mathsf{p}_J \geq 0$ THEN
            compute all sequences of $\mathsf{q}_j - \mathsf{p}_J$ expansions that begin with $J$;
            $\mathcal{T} := \mathcal{T} \cup$ the resulting set of Borel ideals with Hilbert polynomial $\mathsf{q}_j$;
    $\mathcal{S} := \mathcal{T}$;
    $j := j + 1$;
RETURN $\mathcal{S}$

*Proof.* See [Moo12, Algorithm 3.22]. $\square$

**Example 5.13.** To compute all codimension 2 saturated Borel ideals with Hilbert polynomial $\mathsf{p}(t) := 3t + 1$, we first compute $\nabla(\mathsf{p}) = 3$. We produce all length 2 sequences of expansions beginning at $\langle x_0, x_1 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$. The only expandable generator of $\langle x_0, x_1 \rangle$ is $x_1$, with expansion $\langle x_0, x_1^2 \rangle$. Both $x_0$ and $x_1^2$ are expandable in $\langle x_0, x_1^2 \rangle$, with expansions $\langle x_0^2, x_0x_1, x_1^2 \rangle, \langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$. Extending each of these to $\mathbb{K}[x_0, x_1, x_2, x_3]$, their Hilbert polynomials are $3t + 1$ and $3t$, respectively. Thus, we make all possible expansions of $\langle x_0, x_1^3 \rangle$; expansion at $x_0$ gives $\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$, and expansion at $x_1^3$ gives $\langle x_0, x_1^4, x_1^3x_2 \rangle$. Hence, there are three codimension 2 saturated Borel ideals with Hilbert polynomial $3t + 1$ namely, $\langle x_0^2, x_0x_1, x_1^2 \rangle$, $\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$, and $\langle x_0, x_1^4, x_1^3x_2 \rangle$ in $\mathbb{K}[x_0, x_1, x_2, x_3]$.

## 6. KLIMBING TREES

The goal of this section is to understand where Hilbert functions and Hilbert polynomials of saturated Borel ideals coincide. Theorem 6.10 states that among the saturated Borel ideals with a fixed codimension and Hilbert polynomial, the degree of the $K$-polynomial of the lexicographic ideal is *strictly* the largest. The proof tracks the genesis of minimal monomial generators as Algorithm 5.12 traces the path from $1$ to $\mathsf{p}$ in $\mathcal{M}$. Proposition 6.6 identifies where the inequality first occurs, and Proposition 6.8 shows that it persists.

The ***Hilbert series*** of a finitely generated graded $\mathbb{K}[x_0, x_1, \dots, x_n]$-module $M$ is the formal power series $\mathsf{H}_M(T) := \sum_{i\in\mathbb{Z}} \mathsf{h}_M(i)\, T^i \in \mathbb{Z}[T^{-1}][\![T]\!]$. The Hilbert series of $M$ is a rational function $\mathsf{H}_M(T) = (1-T)^{-(n+1)}\,\mathsf{K}_M(T)$, and the ***$K$-polynomial*** of $M$ is the numerator $\mathsf{K}_M$, possibly divisible by $1-T$, of $\mathsf{H}_M$; see [MS05b, Theorem 8.20]. For a quotient by a homogeneous ideal $I$, we use the notation $\mathsf{H}_I := \mathsf{H}_{\mathbb{K}[x_0,x_1,\dots,x_n]/I}$ and $\mathsf{K}_I := \mathsf{K}_{\mathbb{K}[x_0,x_1,\dots,x_n]/I}$.

We consider a fundamental example.

**Example 6.1.** If $S := \mathbb{K}[x_0, x_1, \dots, x_n]$, then we have $\mathsf{K}_S(T) = 1$, as $\mathsf{H}_S(T) = (1-T)^{-(n+1)}$. If $d \in \mathbb{N}$, then we have $\mathsf{H}_{S(-d)}(T) = (1-T)^{-(n+1)}T^d$ and $\mathsf{K}_{S(-d)}(T) = T^d$.

The following well-known lemma is useful.

**Lemma 6.2.** *Let $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$ be a Borel ideal.*

- *(i) We have $\mathsf{K}_I(T) = 1-\sum_g T^{\deg g}(1-T)^{\max g}$, where the sum is over all minimal monomial generators $g$ of $I$, and $\max g$ is the maximum index of the variables dividing $g$.*
- *(ii) We have $\deg \mathsf{K}_I \leq \max_g\{\deg g + \max g\}$, where the maximum is over all the minimal monomial generators $g$ of $I$.*

*Proof.*

- (i) This follows by Lemma 5.2(ii) and Example 6.1; also see [MS05b, Proposition 2.12].
- (ii) This follows immediately from Part (i). □

We apply Lemma 6.2 in an example.

**Example 6.3.** Example 3.4 shows that $L_3^{3t+1} = \langle x_0, x_1^4, x_1^3x_2\rangle$. Lemma 6.2(i) gives

$$\mathsf{K}_{L_3^{3t+1}}(T) = 1 - \left[T^1(1-T)^0 + T^4(1-T)^1 + T^4(1-T)^2\right] = 1 - T - 2T^4 + 3T^5 - T^6.$$

Therefore, $\deg \mathsf{K}_{L_3^{3t+1}} = 6 = \max\{1+0, 4+1, 4+2\}$.

Lemma 6.4 establishes that $\deg \mathsf{H}_I := (\deg \mathsf{K}_I) - (n+1)$ is the maximal value where $\mathsf{h}_I$ and $\mathsf{p}_I$ do not coincide.

**Lemma 6.4.** *Let $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$ be a homogeneous ideal with rational Hilbert series $\mathsf{H}_I(T) = \sum_{i\in\mathbb{Z}} \mathsf{h}_I(i)\, T^i = \mathsf{K}_I(T)(1-T)^{-(n+1)}$ and Hilbert polynomial $\mathsf{p}_I$. We have $\mathsf{h}_I(i) = \mathsf{p}_I(i)$ for all $i > \deg \mathsf{H}_I$, while $\mathsf{h}_I(i) \neq \mathsf{p}_I(i)$ for $i = \deg \mathsf{H}_I$.*

*Proof.* This is straightforward; see [Kem11, Corollary 11.10] for the filtered affine case. □

**Example 6.5.** Example 6.3 shows that $\mathsf{K}_{L_3^{3t+1}}(T) = 1-T-2T^4+3T^5-T^6$, so $\deg \mathsf{H}_{L_3^{3t+1}} = 2$. Lemma 6.4 implies that $\mathsf{h}_{L_3^{3t+1}}(2) \neq \mathsf{p}_{L_3^{3t+1}}(2)$, but that $\mathsf{h}_{L_3^{3t+1}}(i) = \mathsf{p}_{L_3^{3t+1}}(i)$, for all $i > 2$. Indeed, we compute $\mathsf{h}_{L_3^{3t+1}}(\mathbb{N}) = (1, 3, 6, 10, 13, 16, \dots)$ and $\mathsf{p}_{L_3^{3t+1}}(\mathbb{N}) = (1, 4, 7, 10, 13, 16, \dots)$.

The next two propositions capture the behaviour of $\deg \mathsf{K}_I$, for saturated Borel ideals $I$.

**Proposition 6.6.** *Let $L_n^{\mathsf{p}} \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ be any lexicographic ideal.*

*(i) If $m \in L_n^{\mathsf{p}}$ is a minimal monomial generator, then $m$ is expandable if and only if $m$ is the smallest minimal monomial generator of its degree in $L_n^{\mathsf{p}}$.*
*(ii) Let $m_{n-(\ell+1)}$ denote the last minimal monomial generator of $L_n^{\mathsf{p}}$. If $m \neq m_{n-(\ell+1)}$ is any other expandable generator of $L_n^{\mathsf{p}}$, and $(L_n^{\mathsf{p}})'$ is the expansion of $L_n^{\mathsf{p}}$ at $m$, then every minimal monomial generator $g \in (L^{\mathsf{p}})'_n$ satisfies $\deg g < 1 + \deg m_{n-(\ell+1)}$.*
*(iii) Moreover, in Part (ii), we have $\deg \mathsf{K}_{L_n^{\Psi(\mathsf{p})}} > \deg \mathsf{K}_{(L_n^{\mathsf{p}})'}$.*

*Proof.*

(i) Let $L_n^{\mathsf{p}} = \langle m_1, m_2, \ldots, m_{n-(\ell+1)} \rangle$, as in Lemma 3.3. If we have $\deg m_j = \deg m_{j+1}$, then $m_j$ is not expandable, by inspection.

(ii) By Part (i), we have $\deg m < \deg m_{n-(\ell+1)}$. But the minimal generators of $(L_n^{\mathsf{p}})'$ are

$$(L_n^{\mathsf{p}})' = \langle \left( \{m_1, m_2, \ldots, m_{n-(\ell+1)}\} \setminus \{m\} \right) \cup \left\{ m x_{\max m}, m x_{\max m+1}, \ldots, m x_{n-1} \right\} \rangle,$$

and $\deg m_j$ is maximized at $j = n-(\ell+1)$, which gives the desired inequality.

(iii) Example 5.4 shows that $L_n^{\Psi(\mathsf{p})}$ is the expansion of $L_n^{\mathsf{p}}$ at $m_{n-(\ell+1)}$, so that

$$L_n^{\Psi(\mathsf{p})} = \langle m_1, m_2, \ldots, m_{n-(\ell+2)}, m_{n-(\ell+1)} x_{n-(\ell+2)}, m_{n-(\ell+1)} x_{n-(\ell+1)}, \ldots, m_{n-(\ell+1)} x_{n-1} \rangle.$$

Because $\deg m_j$ is maximized at $m_{n-(\ell+1)}$, we have $\deg \mathsf{K}_{L_n^{\Psi(\mathsf{p})}} = \deg m_{n-(\ell+1)} + n$. On the other hand, Lemma 6.2(ii) and Part (ii) yield

$$\begin{aligned} \deg \mathsf{K}_{(L_n^{\mathsf{p}})'} &\leq \max \{\deg g + \max g \mid g \text{ is a minimal monomial generator of } (L_n^{\mathsf{p}})'\} \\ &< 1 + \deg m_{n-(\ell+1)} + n - 1 = \deg \mathsf{K}_{L_n^{\Psi(\mathsf{p})}}, \end{aligned}$$

as desired. □

We demonstrate Proposition 6.6 in an example.

**Example 6.7.** Consider $L_3^{3t} = \langle x_0, x_1^3 \rangle$. Both $x_0$ and $x_1^3$ are expandable by Proposition 6.6(i). Example 3.6 shows that expansion at $x_1^3$ yields $L_3^{3t+1}$, and $\deg \mathsf{K}_{L_3^{3t+1}} = 6$ by Example 6.3. Expansion at $x_0$ yields $(L_3^{3t})' := \langle x_0^2, x_0 x_1, x_0 x_2, x_1^3 \rangle$ and Lemma 6.2(i) gives

$$\mathsf{K}_{(L_3^{3t})'}(T) = 1 - \left[ T^2(1-T)^0 + T^2(1-T)^1 + T^2(1-T)^2 + T^3(1-T)^1 \right] = 1 - 3T^2 + 2T^3.$$

Hence, we have $\deg \mathsf{K}_{(L_3^{3t})'} = 3 < 6$ or equivalently $\deg \mathsf{H}_{(L_3^{3t})'} = -1 < 2 = \deg \mathsf{H}_{L_3^{3t+1}}$.

Proposition 6.8 explains the persistence of the inequality in Proposition 6.6.

**Proposition 6.8.** *Let $I \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$ be a saturated Borel ideal, let $\mathsf{p} := \mathsf{p}_I$, and let $m_{n-(\ell+1)}$ be the last minimal generator of $L_n^{\mathsf{p}}$. Consider the following condition on $I$:*

($\star$) *all minimal generators $g \in I$ satisfy $\deg g < \deg m_{n-(\ell+1)}$ and $\max g \leq \max m_{n-(\ell+1)}$.*

*If $I$ satisfies ($\star$), then the following are true:*

*(i) $\deg \mathsf{K}_{L_n^{\mathsf{p}}} > \deg \mathsf{K}_I$, or equivalently, $\deg \mathsf{H}_{L_n^{\mathsf{p}}} > \deg \mathsf{H}_I$;*
*(ii) if $I'$ denotes any expansion of $I$, then $I'$ satisfies ($\star$) with respect to $L_n^{\Psi(\mathsf{p})}$;*
*(iii) the extension $\Phi(I)$ satisfies ($\star$) with respect to $L_{n+1}^{\mathsf{p}_{\Phi(I)}}$; and*
*(iv) if $I_{(0)}, I_{(1)}, \ldots, I_{(i)}$ is any finite binary sequence of expansions and extensions such that $I_{(0)} := I$, then $I_{(i)}$ satisfies ($\star$).*

*Proof.*

(i) Lemma 6.2(ii) and the condition ($\star$) give

$$\begin{aligned}\deg \mathsf{K}_I &\le \max\{\deg g + \max g \mid g \text{ is a minimal monomial generator of } I\}\\ &< \deg m_{n-(\ell+1)} + \max m_{n-(\ell+1)} = \mathsf{K}_{L_n^{\mathsf{p}}}.\end{aligned}$$

(ii) The condition ($\star$) for the expansion $I'$ becomes that every minimal generator $g' \in I'$ satisfies $\deg g' < 1 + \deg m_{n-(\ell+1)}$ and $\max g' \le n-1$. Both inequalities hold, by definition of the minimal monomial generators of $I'$, and because $I$ satisfies ($\star$).

(iii) An analogous condition to ($\star$) holds between $\Phi(I)$ and $L_n^{\Phi(\mathsf{p})}$. Replacing $L_n^{\Phi(\mathsf{p})}$ by $L_n^{\Psi^j\,\Phi(\mathsf{p})}$, where $j$ is defined by Lemma 5.9, results in higher degree and maximum index of the last minimal generator of $L_n^{\Psi^j\,\Phi(\mathsf{p})}$. Hence, $\Phi(I)$ satisfies ($\star$).

(iv) We apply induction to the length $i$ of the sequence $I_{(0)}, I_{(1)}, \dots, I_{(i)}$. Parts (ii),(iii) resolve the case $i = 1$. If $i > 1$, then Parts (ii),(iii) ensure that $I_{(1)}$ satisfies ($\star$), and we conclude by applying the induction hypothesis to $I_{(1)}, I_{(2)}, \dots, I_{(i)}$. □

**Example 6.9.** The ideal $I := (L_3^{3t})' = \langle x_0^2, x_0x_1, x_0x_2, x_1^3\rangle$ from Example 6.7 satisfies ($\star$) with respect to $L_3^{3t+1} = \langle x_0, x_1^4, x_1^3x_2\rangle$. The expandable monomials in $I$ are $x_0x_2$ and $x_1^3$, with expansions $\langle x_0^2, x_0x_1, x_0x_2^2, x_1^3\rangle$ and $\langle x_0^2, x_0x_1, x_0x_2, x_1^4, x_1^3x_2\rangle$. Both expansions satisfy ($\star$) with respect to their lexicographic ideal $\langle x_0, x_1^4, x_1^3x_2^2\rangle$.

We have $\mathsf{p}_{\Phi(I)}(t) = (3/2)t^2 + (5/2)t + 1$ and $\Phi(3t+1) = (3/2)t^2 + (5/2)t - 1$, so lex-expanding $L_4^{(3/2)t^2+(5/2)t-1} = \langle x_0, x_1^4, x_1^3x_2\rangle$ twice gives $L_4^{\mathsf{p}_{\Phi(I)}} = \langle x_0, x_1^4, x_1^3x_2^2, x_1^3x_2x_3^2\rangle$. Thus, $\Phi(I) = \langle x_0^2, x_0x_1, x_0x_2, x_1^3\rangle \subset \mathbb{K}[x_0, x_1, \dots, x_4]$ also satisfies ($\star$).

We apply Proposition 6.6 and Proposition 6.8 and prove the main result of this section.

**Theorem 6.10.** *Let $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ be a saturated Borel ideal, and let $\mathsf{p} := \mathsf{p}_I$. If $I \neq L_n^{\mathsf{p}}$, then we have $\deg \mathsf{K}_{L_n^{\mathsf{p}}} > \deg \mathsf{K}_I$, or equivalently, $\deg \mathsf{H}_{L_n^{\mathsf{p}}} > \deg \mathsf{H}_I$.*

*Proof.* Both $I$ and $L_n^{\mathsf{p}}$ are saturated and Borel, so are generated by Algorithm 5.12. Let their codimension be $c$, so there is a finite binary sequence of expansions and extensions $I_{(0)}, I_{(1)}, \dots, I_{(i)}$ such that $I_{(0)} = \langle x_0, x_1, \dots, x_{c-1}\rangle \subset \mathbb{K}[x_0, x_1, \dots, x_c]$ and $I_{(i)} = I$. Theorem 3.9 implies that if $I \neq L_n^{\mathsf{p}}$, then there is some $1 \le k \le i$ such that $I_{(j)}$ equals either $\Psi(I_{(j-1)})$ or $\Phi(I_{(j-1)})$, for all $1 \le j \le k-1$, but $I_{(k)} \neq \Psi(I_{(k-1)})$ and $I_{(k)} \neq \Phi(I_{(k-1)})$. By Proposition 6.6(i), $I_{(k)}$ is the expansion of $I_{(k-1)}$ at a minimal generator of nonmaximal degree. Proposition 6.6(ii) then shows that $I_{(k)}$ satisfies the condition ($\star$) from Proposition 6.8. Applying Proposition 6.8(iv) to the subsequence $I_{(k)}, I_{(k+1)}, \dots, I_{(i)}$ shows that $I_{(i)} = I$ satisfies ($\star$), hence, applying Proposition 6.8(i) finishes the proof. □

*Proof of Theorem 1.3.* Theorem 6.10 proves the claim. □

**Example 6.11.** Example 5.13 shows that the saturated Borel ideals in $\mathbb{K}[x_0, x_1, x_2, x_3]$ with Hilbert polynomial $3t+1$ are $\langle x_0^2, x_0x_1, x_1^2\rangle$, $\langle x_0^2, x_0x_1, x_0x_2, x_1^3\rangle$, and $L_3^{3t+1} = \langle x_0, x_1^4, x_1^3x_2\rangle$. Example 6.3 shows that $\deg \mathsf{K}_{L_3^{3t+1}} = 6$, Example 6.7 gives $\deg \mathsf{K}_{\langle x_0^2, x_0x_1, x_0x_2, x_1^3\rangle} = 3$, and Lemma 6.2(i) yields $\deg \mathsf{K}_{\langle x_0^2, x_0x_1, x_1^2\rangle} = 3$.

We reformulate Theorem 6.10 in terms of Hilbert functions using Lemma 6.4.

**Corollary 6.12.** *Let $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ be a saturated Borel ideal, and let $\mathsf{p} := \mathsf{p}_I$. If $I \neq L_n^{\mathsf{p}}$, then there exists $k \in \mathbb{Z}$ such that $\mathsf{h}_I(j) = \mathsf{p}(j)$, for all $j \ge k$, but $\mathsf{h}_{L_n^{\mathsf{p}}}(k) \neq \mathsf{p}(k)$.*

*Proof.* Lemma 6.4 and Theorem 6.10 show that this is the case for $k := 1 + \deg \mathsf{H}_I$. □

## 7. Irreducibility of Random Hilbert Schemes

We finally consider numbers of irreducible components of Hilbert schemes as outcomes of a random variable. Our challenge is to estimate the likelihood that the random variable equals $1$; Theorem 7.8 provides the lower bound $0.5$. We use Theorem 6.10 to prove Theorem 7.4, which implies that all nodes in the Hilbert forest have at least one child corresponding to an irreducible and nonsingular Hilbert scheme. This classification of Hilbert schemes with unique Borel ideals generalizes the result [Got89, Proposition 1].

The next two lemmas are used to prove Theorem 7.4.

**Lemma 7.1.** *Let $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ be a homogeneous ideal, and let $\mathsf{p} := \mathsf{p}_I$.*

*(i) We have $\mathsf{h}_I(i) \geq \mathsf{h}_{L_n^{\mathsf{p}}}(i)$, for all $i \in \mathbb{Z}$, where $L_n^{\mathsf{p}}$ is the corresponding lexicographic ideal.*
*(ii) The Hilbert function of $\Phi(I) := I \cdot \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ is given by $\mathsf{h}_{\Phi(I)}(i) = \sum_{0 \leq j \leq i} \mathsf{h}_I(j)$.*

*Proof.*

(i) Section 3 defines the lexicographic ideal $L_n^{\mathsf{h}} \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ for $\mathsf{h} := \mathsf{h}_I$. We have

$$\mathsf{h}(i) = \dim_{\mathbb{K}} \mathbb{K}[x_0, x_1, \dots, x_n]_i / (L_n^{\mathsf{h}})_i \geq \dim_{\mathbb{K}} \mathbb{K}[x_0, x_1, \dots, x_n]_i / (L_n^{\mathsf{p}})_i,$$

for all $i \in \mathbb{Z}$, because $L_n^{\mathsf{p}}$ contains $L_n^{\mathsf{h}}$, by definition. Hence, we have $\mathsf{h}(i) \geq \mathsf{h}_{L_n^{\mathsf{p}}}(i)$.

(ii) The homogeneous piece $\left(\Phi(I)\right)_i$ has decomposition

$$\left(\Phi(I)\right)_i = \bigoplus_{j \in \mathbb{N}, j \leq i} I_j \cdot x_{n+1}^{i-j} \subset \bigoplus_{j \in \mathbb{N}, j \leq i} \mathbb{K}[x_0, x_1, \dots, x_n]_j \cdot x_{n+1}^{i-j} = \mathbb{K}[x_0, x_1, \dots, x_{n+1}]_i,$$

and the desired equality follows directly. ☐

**Example 7.2.** Let $I := (L_3^{3t})' = \langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$ be as in Example 6.7. Lemma 6.4 implies $\mathsf{h}_I(\mathbb{N}) = \mathsf{p}_I(\mathbb{N}) = (1, 4, 7, 10, 13, 16, \dots)$, compared with $\mathsf{h}_{L_3^{3t+1}}(\mathbb{N}) = (1, 3, 6, 10, 13, 16, \dots)$. We also have $\mathsf{h}_{\Phi(I)}(\mathbb{N}) = (1, 5, 12, 22, 35, 51, \dots)$ and $\mathsf{h}_{\Phi\left(L_3^{3t+1}\right)}(\mathbb{N}) = (1, 4, 10, 20, 33, 49, \dots)$. The values of $\mathsf{h}_I$ and $\mathsf{h}_{L_3^{3t+1}}$ and Lemma 7.1(ii) imply that $\mathsf{p}_{\Phi(I)} = 2 + \mathsf{p}_{\Phi\left(L_3^{3t+1}\right)} = \mathsf{p}_{\Psi^2\,\Phi\left(L_3^{3t+1}\right)}$.

**Lemma 7.3.** *Let $c > 0$, $\mathsf{p}$ an admissible Hilbert polynomial, and $\Lambda$ a finite binary sequence of $\Phi$'s and $\Psi$'s. The number of expandable minimal monomial generators of $L_{c+\deg \Lambda(\mathsf{p})}^{\Lambda(\mathsf{p})}$ is greater than or equal to the corresponding number for $L_{c+\deg \mathsf{p}}^{\mathsf{p}}$.*

*Proof.* This follows from Proposition 6.6(i) and the definition of expandable. ☐

Example 7.2 gives a saturated Borel ideal $I$ such that $\mathsf{p}_{\Phi(I)} - \Phi(\mathsf{p}_I) > 0$. Theorem 7.4 captures and generalizes this behaviour.

**Theorem 7.4.** *Let $\mathsf{p}(t) = \sum_{j=1}^{r} \binom{t+b_j-(j-1)}{b_j}$, for $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$. The lexicographic ideal is the unique saturated Borel ideal of codimension $c$ with Hilbert polynomial $\mathsf{p}$ if and only if:*

*(i) $c \geq 2$ and either $b_r > 0$ or $r \leq 2$; or*
*(ii) $c = 1$ and either $b_r > 0$, $b_1 = b_r$, or $r - s \leq 2$, where $b_1 = b_2 = \cdots = b_s > b_{s+1} \geq \cdots \geq b_r$.*

*Proof.* We begin with $b_r > 0$ and $c > 0$ arbitrary. Remark 2.11 shows that $\mathsf{p}$ equals $\Phi^{b_r}\,\Psi\,\Phi^{b_{r-1}-b_r}\,\Psi \cdots \Psi\,\Phi^{b_2-b_3}\,\Psi\,\Phi^{b_1-b_2}(1)$, so there exists $\mathsf{q}$ such that $\mathsf{p} = \Phi(\mathsf{q})$. Saturated Borel ideals are generated by Algorithm 5.12. The procedure is recursive and generates the codimension $c$ saturated Borel ideals with Hilbert polynomial $\mathsf{p}$ by extending all codimension $c$ saturated Borel ideals with Hilbert polynomial $\mathsf{q} = \nabla\,\Phi(\mathsf{q})$, and keeping the ideals with Hilbert polynomial $\mathsf{p}$.

By Proposition 3.7, we have $\Phi(L_n^{\mathsf{q}}) = L_{n+1}^{\mathsf{p}} \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$, where $n = c + \deg \mathsf{q}$. It suffices to prove the following statement:

*If $J \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ is a saturated, Borel, nonlexicographic ideal, then $\mathsf{p}_{\Phi(J)} \neq \Phi(\mathsf{p}_J)$.*

Let $I := \Phi(J)$ be the extension of such an ideal, $\mathsf{q} := \mathsf{p}_J$, and $\mathsf{p} := \Phi(\mathsf{q})$. By Lemma 5.9, we must show that $\mathsf{p}_I - \mathsf{p} > 0$. Setting $d_{\mathsf{q}} := \deg \mathsf{H}_{L_n^{\mathsf{q}}}$, we show that $\mathsf{h}_I(i) > \mathsf{h}_{L_{n+1}^{\mathsf{p}}}(i)$, for all integers $i \geq d_{\mathsf{q}}$. Lemma 7.1(ii) implies that

$$\mathsf{h}_I(i) = \sum_{0\leq j\leq i} \mathsf{h}_J(j) = \sum_{0\leq j\leq d_{\mathsf{q}}} \mathsf{h}_J(j) + \sum_{d_{\mathsf{q}}<j\leq i} \mathsf{h}_J(j) \text{ and}$$

$$\mathsf{h}_{L_{n+1}^{\mathsf{p}}}(i) = \sum_{0\leq j\leq i} \mathsf{h}_{L_n^{\mathsf{q}}}(j) = \sum_{0\leq j\leq d_{\mathsf{q}}} \mathsf{h}_{L_n^{\mathsf{q}}}(j) + \sum_{d_{\mathsf{q}}<j\leq i} \mathsf{h}_{L_n^{\mathsf{q}}}(j).$$

Theorem 6.10 implies $\deg \mathsf{H}_J < d_{\mathsf{q}}$, so that $\sum_{d_{\mathsf{q}}<j\leq i} \mathsf{h}_J(j) = \sum_{d_{\mathsf{q}}<j\leq i} \mathsf{q}(j) = \sum_{d_{\mathsf{q}}<j\leq i} \mathsf{h}_{L_n^{\mathsf{q}}}(j)$, by Lemma 6.4. We must prove that $\sum_{0\leq j\leq d_{\mathsf{q}}} \mathsf{h}_J(j) > \sum_{0\leq j\leq d_{\mathsf{q}}} \mathsf{h}_{L_n^{\mathsf{q}}}(j)$. Lemma 7.1(i) guarantees that $\sum_{0\leq j\leq d_{\mathsf{q}}} \mathsf{h}_J(j) \geq \sum_{0\leq j\leq d_{\mathsf{q}}} \mathsf{h}_{L_n^{\mathsf{q}}}(j)$ holds, and strict inequality fails if and only if $\mathsf{h}_J(j) = \mathsf{h}_{L_n^{\mathsf{q}}}(j)$, for all $0 \leq j \leq d_{\mathsf{q}}$. But this contradicts Corollary 6.12, so strict inequality holds, and $\mathsf{p}_I - \mathsf{p} > 0$. Hence, the unique codimension $c$ saturated Borel ideal with Hilbert polynomial $\mathsf{p}$ is the lexicographic ideal $L_{n+1}^{\mathsf{p}}$. This covers the case $b_r > 0$, for all $c > 0$.

To prove the remaining cases, we examine Algorithm 5.12.

(i) Let $c \geq 2$ and $b_r = 0$. If $r = 1$, then $\mathsf{p} = 1$ and uniqueness holds. If $r = 2$, then $\mathsf{p} = \Psi\,\Phi^{b_1}(1)$ and to generate saturated Borel ideals with codimension $c$ and Hilbert polynomial $\mathsf{p}$, we take $b_1$ extensions from $L_c^1 = \langle x_0, x_1, \dots, x_{c-1}\rangle$ followed by one expansion. The only expandable generator is $x_{c-1}$, hence, uniqueness again holds.

Conversely, let $b_r = 0$ and $r \geq 3$. Consider $\Psi\,\Phi^{b_1-b_2}(1)$ and its lexicographic ideal $\langle x_0, x_1, \dots, x_{c-2}, x_{c-1}^2, x_{c-1}x_c, \dots, x_{c-1}x_{c+b_1-b_2-1}\rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{c+b_1-b_2}]$. As $c \geq 2$, this ideal has two expandable generators, $x_{c-2}$ and $x_{c-1}x_{c+b_1-b_2-1}$. Lemma 7.3 shows that $L_{c+b_1}^{\Phi^{b_{r-1}}\,\Psi\,\Phi^{b_{r-2}-b_{r-1}}\,\Psi \cdots \Psi\,\Phi^{b_2-b_3}\,\Psi\,\Phi^{b_1-b_2}(1)}$ has at least two expandable generators, giving distinct saturated Borel ideals with Hilbert polynomial $\mathsf{p}$ and codimension $c$.

(ii) Let $c = 1$ and $b_r = 0$. If $b_1 = b_r$, then $\mathsf{p} = r$ and to generate codimension $1$ saturated Borel ideals with Hilbert polynomial $\mathsf{p}$, we take $r-1$ expansions from $L_1^1 = \langle x_0\rangle \subset \mathbb{K}[x_0, x_1]$; the possibilities are $\langle x_0^2\rangle, \langle x_0^3\rangle, \dots, \langle x_0^r\rangle \subset \mathbb{K}[x_0, x_1]$. Let $b_1 = b_2 = \cdots = b_s > b_{s+1} \geq \cdots \geq b_r$ and $r - s \leq 2$. If $r - s = 1$, then we have $\mathsf{p} = \Psi\,\Phi^{b_{r-1}}\,\Psi^{r-2}(1)$ and we take $b_{r-1}$ extensions of $\langle x_0^{r-1}\rangle \subset \mathbb{K}[x_0, x_1]$, followed by the unique expansion of $\langle x_0^{r-1}\rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{1+b_{r-1}}]$. If $r - s = 2$, then we obtain $\mathsf{p} = \Psi\,\Phi^{b_{r-1}}\,\Psi\,\Phi^{b_{r-2}-b_{r-1}}\,\Psi^{r-3}(1)$. We extend $b_{r-2} - b_{r-1}$ times from $\langle x_0^{r-2}\rangle \subset \mathbb{K}[x_0, x_1]$, we expand to obtain $\langle x_0^{r-1}, x_0^{r-2}x_1, \dots, x_0^{r-2}x_{b_{r-2}-b_{r-1}}\rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{1+b_{r-2}-b_{r-1}}]$, we take $b_{r-1}$ further extensions, and we expand at $x_0^{r-2}x_{b_{r-2}-b_{r-1}}$.

Now let $b_r = 0$, $b_1 > b_r$, and $r - s \geq 3$. As in (i), consider the polynomial $\Psi\,\Phi^{b_{s+1}-b_{s+2}}\,\Psi\,\Phi^{b_s-b_{s+1}}\,\Psi^{s-1}(1)$ obtained from $\mathsf{p}$ by truncation, with lexicographic ideal $\langle x_0^{s+1}, x_0^s x_1, \dots, x_0^s x_{b_s-b_{s+1}-1}, x_0^s x_{b_s-b_{s+1}}^2, x_0^s x_{b_s-b_{s+1}} x_{b_s-b_{s+1}+1}, \dots, x_0^s x_{b_s-b_{s+1}} x_{b_s-b_{s+2}}\rangle$. As $b_s > b_{s+1}$, both $x_0^s x_{b_s-b_{s+1}-1}$ and $x_0^s x_{b_s-b_{s+1}} x_{b_s-b_{s+2}}$ are expandable and Lemma 7.3

shows that $L_{1+b_1}^{\Phi^{b_{r-1}}\,\Psi\,\Phi^{b_{r-2}-b_{r-1}}\,\Psi\cdots\Psi\,\Phi^{b_2-b_3}\,\Psi\,\Phi^{b_1-b_2}(1)}$ has at least two distinct expansions, both with Hilbert polynomial $\mathsf{p}$ and codimension $c = 1$. ☐

**Remark 7.5.** The case $b_r > 0$ follows from Theorem 6.10. Another approach may exist using Stanley decompositions; see [MS05a, SW91, Sta82]. Indeed, Proposition 3.7 follows by considering a Stanley decomposition of the lexicographic ideal, while Lemma 5.2(ii) gives a Stanley decomposition of $I$. We thank D. Maclagan for pointing this out.

*Proof of Theorem 1.1.* Theorem 7.4 proves the claim. ☐

Many well-known Hilbert schemes fit into this classification, and motivated [Got89].

**Example 7.6.** For integers $d, k > 0$, the Hilbert polynomial of a degree $d$ hypersurface in $\mathbb{P}^k$ is $\Phi^k(d) = \sum_{i=0}^{k} \binom{t+i}{i+1} - \binom{t+i-d}{i+1} = \binom{t+k}{k} - \binom{t+k-d}{k}$, by the addition formula. Hilbert schemes $\mathrm{Hilb}^{\Phi^k(d)}(\mathbb{P}^n)$ are irreducible and nonsingular; see [ACG11, Example 2.3] or [Ådl85].

**Lemma 7.7.** *If the lexicographic ideal is the unique saturated Borel ideal with Hilbert polynomial* $\mathsf{p}$ *and codimension* $c$*, then* $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ *is irreducible and nonsingular, where* $n := c + \deg \mathsf{p}$.

*Proof.* Every component and intersection of components of $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ contains a point $[X_I]$ defined by a saturated Borel ideal $I$; see [Ree95, Remark 2.1]. Lexicographic points are nonsingular by [RS97, Theorem 1.4], so $[X_{L_n^{\mathsf{p}}}]$ cannot lie on an intersection of components. Thus, $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ has a unique, generically nonsingular, irreducible component.

Suppose $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ has a singular point, given by $I \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$. For $G \in \mathrm{GL}_{n+1}(\mathbb{K})$, the point $[X_{G\cdot I}] \in \mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ is also singular, and for generic $G \in \mathrm{GL}_n(\mathbb{K})$, the initial ideal of $G \cdot I$ with respect to any monomial ordering is saturated and Borel; see [Gal74, Theorem 2], [BS87b, Proposition 1]. Thus, a one-parameter family of singular points degenerating to the lexicographic ideal exists; see [Bay82, Proposition I.2.12], [Eis95, Theorem 15.17]. By upper semicontinuity of cohomology of the normal sheaf, the lexicographic ideal is singular, a contradiction; see [Har77, Theorem III.12.8], [Har10, Theorem 1.1(b)], [RS97, Theorem 1.4]. Hence, $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ is nonsingular and irreducible. ☐

Interpreting the set of Hilbert schemes as a probability space, we see that at least half the vertices at any height correspond to irreducible, nonsingular Hilbert schemes.

**Theorem 7.8.** *Let* $\mathrm{irr}\colon \mathcal{H} \to \mathbb{N}$ *be the random variable taking a Hilbert scheme to its number of irreducible components, and for positive* $c \in \mathbb{Z}$*, let* $\mathrm{irr}_c := \mathrm{irr}|_{\mathcal{H}_c}$*. We have* $\Pr(\mathrm{irr} = 1) > 0.5$ *and* $\Pr(\mathrm{irr}_c = 1) > 0.5$*, for all positive* $c \in \mathbb{Z}$.

*Proof.* We have $\Pr(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)) := f(c) f_c(k)/2^k$ for $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n) \in \mathcal{H}_c$ at height $k$, where $f$ and $f_c$ are normalized functions as in Section 4. Let $A$ be the set of irreducible Hilbert schemes. We compute

$$\begin{aligned}
\Pr(\mathrm{irr} = 1) = \sum_{\substack{\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n) \\ \text{in } A}} \Pr(\{\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)\}) &\geq \sum_{c>0} \left( f(c) f_c(0) + \sum_{k \geq 1} \frac{2^k}{2} \frac{f(c) f_c(k)}{2^k} \right) \\
&= \sum_{c>0} \left( \frac{f(c) f_c(0)}{2} + \sum_{k \in \mathbb{N}} \frac{f(c) f_c(k)}{2} \right) \\
&= \sum_{c>0} \left( \frac{f(c) f_c(0)}{2} + \frac{f(c)}{2} \right) = \sum_{c>0} \left( \frac{f(c) f_c(0)}{2} \right) + \frac{1}{2},
\end{aligned}$$

as in each tree $\mathcal{H}_c$, at height $k \geq 1$, there are at least $2^k/2$ vertices corresponding to irreducible Hilbert schemes, by Theorem 7.4 and Corollary 7.7.

In a chosen Hilbert tree $\mathcal{H}_c$, we similarly compute

$$\Pr\big(\mathrm{irr}_c = 1\big) = \sum_{\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n) \in A \cap \mathcal{H}_c} \Pr\big(\{\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)\}\big) \geq f_c(0) + \sum_{k \geq 1} \frac{2^k}{2} \frac{f_c(k)}{2^k} = \frac{f_c(0)}{2} + \sum_{k \in \mathbb{N}} \frac{f_c(k)}{2} > \frac{1}{2}.$$

Hence, the probability that a random Hilbert scheme is irreducible is greater than $0.5$. □

*Proof of Theorem 1.2.* Theorem 7.8 proves the claim. □

## References


[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris. MR 2807457

[Ådl85] Bjørn Ådlandsvik, *Hilbert schemes of hypersurfaces and numerical criterions*, Math. Scand. **56** (1985), no. 2, 163–170. MR 813634

[Bay82] David Allen Bayer, *The division algorithm and the Hilbert scheme*, ProQuest LLC, Ann Arbor, MI, 1982, Thesis (Ph.D.)–Harvard University. MR 2632095

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956 (95h:13020)

[Bil95] Patrick Billingsley, *Probability and measure*, third ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1995, A Wiley-Interscience Publication. MR 1324786

[BS87a] David Bayer and Michael Stillman, *A criterion for detecting $m$-regularity*, Invent. Math. **87** (1987), no. 1, 1–11. MR 862710 (87k:13019)

[BS87b] ———, *A theorem on refining division orders by the reverse lexicographic order*, Duke Math. J. **55** (1987), no. 2, 321–328. MR 894583

[CLMR11] Francesca Cioffi, Paolo Lella, Maria Grazia Marinari, and Margherita Roggero, *Segments and Hilbert schemes of points*, Discrete Math. **311** (2011), no. 20, 2238–2252. MR 2825669 (2012j:14004)

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960

[Fog68] John Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math **90** (1968), 511–521. MR 0237496

[FP96] Barbara Fantechi and Rita Pardini, *On the Hilbert scheme of curves in higher-dimensional projective space*, Manuscripta Math. **90** (1996), no. 1, 1–15. MR 1387750

[Gal74] André Galligo, *À propos du théorème de-préparation de Weierstrass*, Fonctions de plusieurs variables complexes (Sém. François Norguet, octobre 1970–décembre 1973; à la mémoire d'André Martineau), Springer, Berlin, 1974, Thèse de 3ème cycle soutenue le 16 mai 1973 à l'Institut de Mathématique et Sciences Physiques de l'Université de Nice, pp. 543–579. Lecture Notes in Math., Vol. 409. MR 0402102

[GKP94] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete mathematics*, second ed., Addison-Wesley Publishing Company, Reading, MA, 1994, A foundation for computer science. MR 1397498

[Got78] Gerd Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), no. 1, 61–70. MR 0480478 (58 #641)

[Got89] ———, *Some Irreducible Hilbert Schemes*, Math. Z. **201** (1989), no. 1, 13–17. MR 990184

[Har66] Robin Hartshorne, *Connectedness of the Hilbert scheme*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 5–48. MR 0213368

[Har77] ———, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[Har10] ______, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010. MR 2583634

[Iar72] A. Iarrobino, *Reducibility of the families of* 0*-dimensional schemes on a variety*, Invent. Math. **15** (1972), 72–77. MR 0301010

[IK99] Anthony Iarrobino and Vassil Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman. MR 1735271

[Kem11] Gregor Kemper, *A course in commutative algebra*, Graduate Texts in Mathematics, vol. 256, Springer, Heidelberg, 2011. MR 2766370

[Mac27] F. S. Macaulay, *Some Properties of Enumeration in the Theory of Modular Systems*, Proc. London Math. Soc. **S2-26** (1927), no. 1, 531. MR 1576950

[Moo12] Dennis Moore, *Hilbert polynomials and strongly stable ideals*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–University of Kentucky. MR 3211660

[MS05a] Diane Maclagan and Gregory G. Smith, *Uniform bounds on multigraded regularity*, J. Algebraic Geom. **14** (2005), no. 1, 137–164. MR 2092129

[MS05b] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR 2110098 (2006d:13001)

[Mum62] David Mumford, *Further pathologies in algebraic geometry*, Amer. J. Math. **84** (1962), 642–648. MR 0148670

[PS85] Ragni Piene and Michael Schlessinger, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. **107** (1985), no. 4, 761–774. MR 796901

[PS05] Irena Peeva and Mike Stillman, *Connectedness of Hilbert schemes*, J. Algebraic Geom. **14** (2005), no. 2, 193–211. MR 2123227

[Ree92] Alyson April Reeves, *Combinatorial structure on the Hilbert scheme*, ProQuest LLC, Ann Arbor, MI, 1992, Thesis (Ph.D.)–Cornell University. MR 2688239

[Ree95] Alyson A. Reeves, *The radius of the Hilbert scheme*, J. Algebraic Geom. **4** (1995), no. 4, 639–657. MR 1339842

[RS97] Alyson Reeves and Mike Stillman, *Smoothness of the lexicographic point*, J. Algebraic Geom. **6** (1997), no. 2, 235–246. MR 1489114 (98m:14003)

[Sta82] Richard P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math. **68** (1982), no. 2, 175–193. MR 666158

[Sta12] ______, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112

[SW91] Bernd Sturmfels and Neil White, *Computing combinatorial decompositions of rings*, Combinatorica **11** (1991), no. 3, 275–293. MR 1122013

[Vak06] Ravi Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164** (2006), no. 3, 569–590. MR 2227692

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