COMPUTATIONAL DEFORMATION THEORY OF PROJECTIVE SCHEMES

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Abstract. A Hilbert scheme is a parameter space for all subschemes of projective space with a fixed Hilbert polynomial. Hilbert schemes are fundamental moduli spaces, whose local geometry is studied via the deformation theory of projective schemes. We give a concise introduction to deformation theory, computing specific examples of the cotangent cohomology of projective schemes. We then give a detailed account of the *power series ansatz*, a procedure for computing versal pairs of local moduli functors, and studying the local geometry of Hilbert schemes. This is explained through two concrete examples. We end with some open questions about Hilbert schemes and research goals.

1. Hilbert Schemes

We begin with a description of the most basic type of moduli space in algebraic geometry, namely, Hilbert schemes of projective space over a field. A Hilbert scheme parametrizes all of the closed subschemes with a fixed Hilbert polynomial of a fixed projective space. Although they are fundamental, relatively little is known about the geometry of Hilbert schemes in general. Hartshorne proves in [9] that Hilbert schemes are connected, but beyond this general results are scarce. According to Murphy's Law for Hilbert schemes (see [21]), we should not expect arbitrary Hilbert schemes to have simple geometry. However, there seems to be a gap in the geography of Hilbert schemes, between the known examples and the known pathologies. It is in this gap that we intend to work. We introduce some notation, and then describe some of what is known about the geometry of Hilbert schemes.

Let $\mathbb{F}$ denote an algebraically closed field of characteristic zero, and let $\mathbb{P}^n$ be the projective space of dimension n over $\mathbb{F}$. Let X denote a closed subscheme of $\mathbb{P}^n$ with Hilbert polynomial p. If X' is another subscheme of $\mathbb{P}^n$ with Hilbert polynomial p, we say that X and X' have the same *type*. The *Hilbert scheme of subschemes of* $\mathbb{P}^n$ *of type* X parametrizes all subschemes of $\mathbb{P}^n$ whose Hilbert polynomial equals the Hilbert polynomial of X. The Hilbert scheme is a projective scheme over $\mathbb{F}$, whose closed points correspond bijectively to the subschemes of $\mathbb{P}^n$ of type X. Continuously varying a point of the Hilbert scheme corresponds to continuously deforming subschemes of type X in $\mathbb{P}^n$. For example, one-parameter families of subschemes of type X correspond to curves in the Hilbert scheme.

The scheme-theoretic version of continuously varying families is given by flat morphisms; flatness ensures that the fibres of a morphism of schemes are closely related. A morphism of schemes $\varphi\colon \mathcal{X} \to B$ is *flat at* $x \in \mathcal{X}$ if $\mathscr{O}_{\mathcal{X},x}$ is a flat $\mathscr{O}_{B,\varphi(x)}$-module via the local homomorphism $\varphi_x^\sharp$, and is *flat* if it is flat at all $x \in \mathcal{X}$ (see [10, p. 254]). The following theorem of Grothendieck ([6], [8, p. 6], [11, p. 5–6]) precisely describes the mentioned correspondence.

Theorem 1.1. *Let $X \subseteq \mathbb{P}^n$ be a closed subscheme with Hilbert polynomial p. There exists a projective scheme $\mathcal{H}_p^n$, and a subscheme $\mathcal{U}_p^n \subseteq \mathbb{P}^n_{\mathcal{H}_p^n} = \mathbb{P}^n \times_{\mathbb{F}} \mathcal{H}_p^n$ flat over $\mathcal{H}_p^n$, such that the set of fibres of $\mathcal{U}_p^n$ over closed points equals the set of subschemes of $\mathbb{P}^n$ of type X. Moreover, if $\mathcal{X} \subseteq \mathbb{P}^n_B$ is any flat family over an $\mathbb{F}$-scheme B such that every closed fibre of $\mathcal{X}$ has Hilbert polynomial p, then there exists a unique morphism $B \to \mathcal{H}_p^n$ such that $\mathcal{X} = B \times_{\mathcal{H}_p^n} \mathcal{U}_p^n$.*

The techniques used to establish the existence of Hilbert schemes are somewhat removed from the tools used to study them. Hilbert schemes are mainly studied via their universal property.

Let us consider some concrete examples of Hilbert schemes.

Example 1.1. Let $X \subseteq \mathbb{P}^2$ be a degree δ plane curve, in other words, X is the vanishing locus of a single homogeneous polynomial $f \in B = \mathbb{F}[x,y,z]$ of degree δ. The defining equation f lies in B_δ, the $\mathbb{F}$-span of the $\binom{2+\delta}{\delta}$ monomials of degree δ. Thus, $f = \sum_{i=0}^N c_i x^{\alpha_i}$, where $N = \binom{2+\delta}{\delta} - 1$ and x^{α_i} is the i^{th} monomial (after putting them in some order). There is a bijective correspondence between curves f and coordinates $[c_0 : c_1 : \cdots : c_N] \in \mathbb{P}^N = \mathbb{P}(B_\delta)$, because $X = V(f)$ is unchanged by scaling f.

Consider the bihomogeneous polynomial $F = \sum_{i=0}^N a_i x^{\alpha_i}$, where $a_0, a_1, \ldots, a_N$ are coordinates on $\mathbb{P}^N$. It defines a closed subscheme $\mathcal{C} = V(F) \subseteq \mathbb{P}^2_{\mathbb{P}^N} = \mathbb{P}^2 \times_{\mathbb{F}} \mathbb{P}^N$ such that the fibre of $\mathcal{C}$ over a closed point $c = [c_0 : c_1 : \cdots : c_N] \in \mathbb{P}^N$ is the curve $V(f)$ corresponding to c. In fact, it can be shown that $\mathcal{H}_p^2 = \mathbb{P}^N$ and $\mathcal{U}_p^n = \mathcal{C}$, where $p(t) = \delta t - \frac{\delta(\delta-3)}{2}$ is the Hilbert polynomial of X.

Example 1.2. Let $X \subseteq \mathbb{P}^3$ be a *twisted cubic* curve. All such curves are projectively equivalent to the *rational normal curve* of degree 3, i.e. the image of the *Veronese embedding* $\nu_3 \colon \mathbb{P}^1 \to \mathbb{P}^3$ defined by $[u : v] \mapsto [u^3 : u^2v : uv^2 : v^3]$. The space of twisted cubics is identified with the 12-dimensional quotient $H_0 = SL_4(\mathbb{F})/SL_2(\mathbb{F})$. Since twisted cubics have degree 3 and (arithmetic) genus 0, their Hilbert polynomial is $p(t) = 3t+1$. However, any disjoint union of a plane cubic $X' \subseteq \mathbb{P}^2 \subseteq \mathbb{P}^3$ with a reduced point in $\mathbb{P}^3$ also has Hilbert polynomial $3t+1$. The space H_0' of such "degenerate twisted cubics" is 15. Considering $H_0, H_0' \subseteq \mathcal{H}^3_{3t+1}$ as subschemes, let $H, H' \subseteq \mathcal{H}^3_{3t+1}$ denote their topological closures. Piene and Schlessinger prove in [17] that $\mathcal{H}^3_{3t+1} = H \cup H'$, where H, H' are nonsingular, rational, and intersect transversally in a nonsingular, rational, 11-dimensional space.

The second example shows that Hilbert schemes can be reducible, even when they parametrize "nice" objects in a low-dimensional projective space. Hilbert schemes are generally high-dimensional, but, due to their universal property, they can be studied through the geometry of the subschemes they parametrize. The following theorem tells us how to compute the *Zariski tangent space* $T_{\mathcal{H}_p^n,[X]}$ of $\mathcal{H}_p^n$ at the point $[X]$ associated to X.

Theorem 1.2. *Let $X \subseteq \mathbb{P}^n$ be a closed subscheme with Hilbert polynomial p, and let $\mathcal{H}_p^n$ be the associated Hilbert scheme. The Zariski tangent space of $\mathcal{H}_p^n$ at $[X]$ is given by $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$ where $\mathscr{N}_{X/\mathbb{P}^n}$ is the normal sheaf of X in $\mathbb{P}^n$.*

Proof. Let $D = \mathbb{F}[\varepsilon] \cong \mathbb{F}[u]/(u^2)$ denote the ring of *dual numbers*. The Zariski tangent space to $\mathcal{H}_p^n$ at $[X]$ is given by the set of morphisms $\mathrm{Spec}(D) \to \mathcal{H}_p^n$ such that $(\varepsilon) \mapsto [X]$ (see [3, p. 256–8]). Any such morphism corresponds uniquely to a family over $\mathrm{Spec}(D)$ by pullback of $\mathcal{U}_p^n$. Pulling this family back along the canonical map $\mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(D)$ yields the trivial family $X \to \mathrm{Spec}(\mathbb{F})$. In other words, the Zariski tangent space is given by the set of families $\mathcal{X} \to \mathrm{Spec}(D)$ whose closed fibre is $X \to \mathrm{Spec}(\mathbb{F})$.

Let $I_X \subseteq B = \mathbb{F}[x_0, \ldots, x_n]$ be the saturated graded ideal defining X. A closed subscheme $\mathcal{X} \subseteq \mathbb{P}^n_D$ is uniquely determined by a saturated graded ideal $I' \subseteq B' = D[x_0, \ldots, x_n]$, and since flatness is a local condition we work locally in $(B'/I')[\overline{f}^{-1}]_0 = B'[f^{-1}]_0/I'[f^{-1}]_0$, for $f \in B'_+$ (of positive degree). The local criterion of flatness used in [11, p. 10] implies that a D-module M' is flat if and only if $M = M' \otimes_D \mathbb{F}$ is flat over $\mathbb{F}$ and the natural homomorphism $M \otimes_{\mathbb{F}} (\varepsilon) \to M'$ is injective. Applying this to $M' = B'[f^{-1}]_0/I'[f^{-1}]_0$ shows that $\mathcal{X}$ is locally flat over $\mathrm{Spec}(D)$ (in the chart $\mathrm{Spec}\,(B'[f^{-1}]_0/I'[f^{-1}]_0)$) if and only if $M \otimes_{\mathbb{F}} (\varepsilon) \to M'$ is injective. Proceeding as in [11, p. 11–12] shows that the map is injective if and only if the ideal $I'[f^{-1}]_0$ is determined by a $B[f^{-1}]_0$-linear map $\phi_f \colon I_X[f^{-1}]_0 \to B[f^{-1}]_0/I_X[f^{-1}]_0$. More precisely, we say that $I'[f^{-1}]_0$ is determined by ϕ_f if the ideal $I'[f^{-1}]_0$ is given by $\{a + \varepsilon b \mid a \in I_X[f^{-1}]_0,\ b \in B[f^{-1}]_0,\ \phi_f(a) = \overline{b} \in B[f^{-1}]_0/I_X[f^{-1}]_0\}$.

Gluing this local data implies that $\mathcal{X}$ is flat over D and pulls back to X if and only if on the local charts $\mathrm{Spec}\,(B'[f^{-1}]_0/I'[f^{-1}]_0)$ of $\mathcal{X}$ the ideal $I'[f^{-1}]_0$ is determined by a $B[f^{-1}]_0$-linear map $\phi_f \colon I_X[f^{-1}]_0 \to B[f^{-1}]_0/I_X[f^{-1}]_0$, and for every $g \in B'_+$ the ideal $I'[(fg)^{-1}]_0$ is similarly determined by a $B[(fg)^{-1}]_0$-linear map ϕ_{fg} such that the diagram

$$\begin{array}{ccc} I_X[f^{-1}]_0 & \xrightarrow{\phi_f} & B[f^{-1}]_0/I_X[f^{-1}]_0 \\ \downarrow & & \downarrow \\ I_X[(fg)^{-1}]_0 & \xrightarrow{\phi_{fg}} & B[(fg)^{-1}]_0/I_X[(fg)^{-1}]_0 \end{array}$$

commutes, where the vertical maps are localizations.

Such data defines a global section of the normal sheaf $\mathscr{N}_{X/\mathbb{P}^n} = \mathscr{H}om_{\mathbb{P}^n}(\mathscr{I}_X, \mathscr{O}_X)$, that is, a homomorphism of sheaves of $\mathscr{O}_{\mathbb{P}^n}$-modules $\phi : \mathscr{I}_X \to \mathscr{O}_X$. Indeed, since $\mathscr{I}_X = \widetilde{I_X}$ and $\mathscr{O}_X = \widetilde{B/I_X}$ are the associated sheaves to the graded B-modules I_X and B/I_X, we know that $\mathscr{I}_X(U_f) = I_X[f^{-1}]_0$ and $\mathscr{O}_X(U_f) = B[f^{-1}]_0/I_X[f^{-1}]_0$ are exactly the rings and ideals above, where $U_f = D_+(f)$ is the principal open subscheme of the coordinate f. Commutativity of the maps ϕ_f, ϕ_{fg} with localizations is equivalent to commutativity with the restrictions

$$\begin{array}{ccc} \mathscr{I}_X(U_f) & \xrightarrow{\phi_f} & \mathscr{O}_X(U_f) \\ \downarrow \rho^{U_f}_{U_{fg}} & & \downarrow \rho^{U_f}_{U_{fg}} \\ \mathscr{I}_X(U_{fg}) & \xrightarrow{\phi_{fg}} & \mathscr{O}_X(U_{fg}) \end{array}$$

where U_{fg} equals $U_f \cap U_g = D_+(fg)$. Since the principal open subsets $\{U_f\}$ cover $\mathbb{P}^n$, and since the horizontal maps ϕ_f, ϕ_{fg} are respectively $\mathscr{O}_{\mathbb{P}^n}(U_f)$-linear and $\mathscr{O}_{\mathbb{P}^n}(U_{fg})$-linear, there is a unique morphism of $\mathscr{O}_{\mathbb{P}^n}$-modules determined by the data, as desired.

Conversely, a morphism of $\mathscr{O}_{\mathbb{P}^n}$-modules $\phi : \mathscr{I}_X \to \mathscr{O}_X$ determines compatible homomorphisms as above, which in turn determine a closed subscheme $\mathcal{X} \subseteq \mathbb{P}^n_D$ flat over D that pulls back to X over $\mathbb{F}$. Hence, global sections $\phi \in H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$ correspond uniquely to families $\mathcal{X} \to \mathrm{Spec}(D)$ that pull back to $X \to \mathrm{Spec}(\mathbb{F})$, which in turn determine the Zariski tangent space. ♫

Example 1.3. Let $X \subseteq \mathbb{P}^2$ be the plane conic $X = V(y^2 - xz)$, which is the image of the Veronese embedding $\nu_2 \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ defined by $[u : v] \mapsto [u^2 : uv : v^2]$. Viewing X as a Cartier divisor on $\mathbb{P}^2$, we have $\mathscr{N}_{X/\mathbb{P}^2} = \mathscr{O}_{\mathbb{P}^2}(X)|_X$ (see [10, p. 182]), which equals $\mathscr{O}_{\mathbb{P}^2}(2)|_X = \mathscr{O}_X(2)$, and pulls back to $\mathscr{O}_{\mathbb{P}^1}(4)$ under the embedding. Hence, we obtain $T_{\mathcal{H}^2_p,[X]} = H^0(X, \mathscr{O}_X(2)) = H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(4)) \cong \mathbb{F}^5$, where p is the Hilbert polynomial of X. This agrees with our construction in Example 1.1 of $\mathcal{H}^2_{2t+1} = \mathbb{P}^5$.

Example 1.4. Following [11, p. 92–4], we outline Mumford's second pathological example from [16]. Let $X \subseteq \mathbb{P}^3$ be a nonsingular irreducible curve of degree 14 and genus 24. For example, if we let $S \subseteq \mathbb{P}^3$ be a cubic surface realized as a blowup of $\mathbb{P}^2$ at 6 points in general position, let H be a hyperplane section of S, and let L be an exceptional curve of the projection $\pi \colon S \to \mathbb{P}^2$, then curves of type X are found in the linear series $|4H + 2L|$ on S. Varying both the cubic surface and the curve $X \in |4H + 2L|$ yields a 56-dimensional family H_0 of degree 14, genus 24 curves. However, one computes $h^0(\mathscr{N}_{X/\mathbb{P}^3}) = 57$, and proves that no higher-dimensional family contains H_0. Hence, the closure $\overline{H_0}$ in the Hilbert scheme is a 56-dimensional irreducible component whose general point has a 57-dimensional Zariski tangent space.

Mumford's example shows that Hilbert schemes of relatively low-degree, low-genus curves in $\mathbb{P}^3$ may have generically nonreduced components. Mumford's example is an instance of *Murphy's Law for Hilbert schemes*, formulated in [8, p. 18] as follows: "There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme." A precise interpretation of Murphy's Law is given by Vakil in [21], roughly saying that Hilbert schemes of nice objects can have arbitrary singularities.

Here is an example which shows that Hilbert schemes of small numbers of points in a low-dimensional projective space can be reducible.

Example 1.5. Let $X \subseteq \mathbb{P}^4$ be a subscheme of eight points, i.e. X is a subscheme of length 8 in $\mathbb{P}^4$. The *smoothable component* $H \subseteq \mathcal{H}^4_8$ is the closure of the set of points of $\mathcal{H}^4_8$ corresponding to unions of 8 distinct points in $\mathbb{P}^4$. By counting degrees of freedom, we see that the dimension of H is 32.

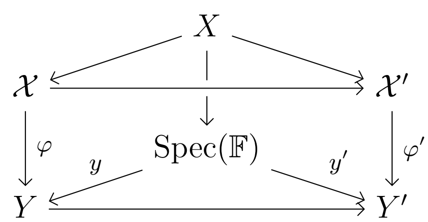
Let $U = D_+(f) \subseteq \mathbb{P}^4$ be an open affine chart of projective space, where f is any linear form on $\mathbb{P}^4$. If $\mathscr{O}_{\mathbb{P}^4}(U) = \mathbb{F}[x, y, z, w]$ is the affine coordinate ring of U and $\mathfrak{m} \subseteq \mathscr{O}_{\mathbb{P}^4}(U)$ is a maximal ideal corresponding to some closed point, then, letting $V \subseteq \mathfrak{m}^2/\mathfrak{m}^3$ be a 7-dimensional vector subspace, and letting I be the ideal generated by V and $\mathfrak{m}^3$, the subscheme $X = \operatorname{Spec}(\mathscr{O}_{\mathbb{P}^4}(U)/I) \subseteq U \subseteq \mathbb{P}^4$ has length 8. Varying the closed point defined by $\mathfrak{m}$ and the subspace V results in a 25-dimensional family of subschemes of eight points in $\mathbb{P}^4$. Further, for the particular example $I = (x^2, xy, y^2, z^2, zw, w^2, xz - yw)$, one directly computes $h^0(\mathscr{N}_{X/\mathbb{P}^4}) = 25$ via the cotangent cohomology methods of the next section. Thus, X cannot lie on the smoothable component, so there exists a *nonsmoothable component* of dimension 25, consisting of *nonsmoothable* fat points of length 8, which do not arise as limits of 8 distinct points in $\mathbb{P}^4$.

Hilbert schemes of points on $\mathbb{P}^2$ are smooth and irreducible by [4] or [11, p. 67]. For at most 8 points, the Hilbert scheme is reducible with two components if and only if $n \geq 4$ by [2], i.e. the example given above is minimal. However, Iarrobino proves in [12] that, for any $n \geq 3$, the Hilbert schemes $\mathcal{H}^n_\ell$ are reducible for all numbers of points $\ell \gg 0$. In particular, he proves that the Hilbert schemes $\mathcal{H}^3_\ell$ are reducible for all $\ell \geq 102$, and improves this bound to 78 in [13]. A recent preprint (see [1]) proves that $\mathcal{H}_\ell^3$ is irreducible for $\ell \leq 10$ points, while the minimal ℓ for which $\mathcal{H}_\ell^3$ has multiple components is not yet known.

2. Basic Deformation Theory

Deformation theory is the study of how geometric objects can be continuously deformed in families. Thus, the deformation theoretic properties of a closed subscheme $X \subseteq \mathbb{P}^n$ reflect the local geometry of the Hilbert scheme near $[X] \in \mathcal{H}_p^n$. In this section, we give some basic definitions and constructions from deformation theory. Particularly important are the cotangent cohomology modules, which tell us about nontrivial first-order deformations and obstructions to lifting deformations. We also define versality, which is studied in more depth in the next section.

Let X and Y be locally finite type, separated $\mathbb{F}$-schemes. A *family of deformations of X over Y* is a surjective, flat morphism $\varphi\colon \mathcal{X} \to Y$ such that there exists a specified $\mathbb{F}$-rational point $y \in Y$ and an isomorphism of X with the *special fibre* $\varphi^{-1}(y)$. We often shorten this to *a family for X over Y*. The scheme Y is called the *parameter* or *base* of the family, and $\mathcal{X}$ is called the *total family*. Closed fibres of φ are called *deformations* of X. We denote such a family for X over Y by (φ, y). A *morphism* $(\varphi, y) \to (\varphi', y')$ between two families for X is a pair of morphisms $\mathcal{X} \to \mathcal{X}', Y \to Y'$ such that the canonical diagram



is commutative. An *isomorphism* of families is a morphism with a two-sided inverse.

A *trivial* family for X over Y is isomorphic to the family $\pi_2\colon X \times_{\mathbb{F}} Y \to Y$ with any choice of $y \in Y$. A *first-order* family for X has base $\operatorname{Spec}(D)$, where $D = \mathbb{F}[\varepsilon]$ is the ring of dual numbers. An *infinitesimal* family for X has base $\operatorname{Spec}(S)$, where S is a local artinian $\mathbb{F}$-algebra with residue field $\mathbb{F}$. Every first-order family is infinitesimal.

If $X \subseteq \mathbb{P}^n$ is a subscheme with Hilbert polynomial p, then the flat families $\mathcal{X} \subseteq \mathbb{P}_B^n$ mentioned in Theorem 1.1, whose closed fibres have Hilbert polynomial p, are called *embedded families of deformations of X in $\mathbb{P}^n$*. Any embedded family for X is a family for X. In order to study embedded families for $X \subseteq \mathbb{P}^n$, we need to define the associated cotangent complex of the embedding, as developed by Lichtenbaum and Schlessinger in [15]. The cohomology of the cotangent complex captures information about deformations and obstructions, which is applied in the power series ansatz in the next section.

We construct the cotangent complex following the exposition of [11, p. 19], but using graded rings and modules. Let A and B be $\mathbb{Z}$-graded rings, M a graded A-module, and suppose that A is a graded B-algebra via a degree zero graded homomorphism $B \to A$. Suppose that A is generated by homogeneous elements $\{a_i\}_{i\in\Lambda}$ as a B-algebra, where Λ is an arbitrary index set. Let $R = B[\{x_i\}_{i\in\Lambda}]$ denote a polynomial ring over the same index.

The ring R is naturally a graded B-algebra via $\deg(x_i) = \deg(a_i)$ for all $i \in \Lambda$. In particular, the canonical B-algebra homomorphism $R \to A$ has degree 0. The kernel I of this map is $\mathbb{Z}$-graded, and there exists a short exact sequence $0 \to I \to R \to A \to 0$ of degree zero homomorphisms of graded B-modules.

Choose homogeneous generators $\{f_j\}_{j\in\Lambda'}$ for I over some index Λ'. Let F denote the free graded R-module of rank $|\Lambda'|$ whose jth generator has degree $\delta_j = \deg(f_j)$. The R-linear map $\phi_0 \colon F \to I$ taking the jth generator to f_j is graded of degree zero, thus has graded kernel, say Q, giving a short exact sequence

$$0 \longrightarrow Q \longrightarrow F = \bigoplus_{j\in\Lambda'} R(-\delta_j) \xrightarrow{\phi_0} I \longrightarrow 0$$

of degree zero homomorphisms of graded R-modules. This constructs the first term $F_0 = F$ of a free resolution of I, where Q is the module of *first syzygies* (i.e. relations) between the generators determined by F. The *Koszul relations* in F are the elements of the form $\phi_0(m)m' - \phi_0(m')m$, which generate a graded submodule $\operatorname{Kos}(F)$ of F that is clearly contained in Q.

Set $L_2 := Q/\operatorname{Kos}(F)$, which is *a priori* an R-module, but is also easily seen to be a graded A-module. Indeed, given $f \in I$ and $q \in Q$, we have $f = \phi_0(f')$ for some $f' \in F$, so $fq = \phi_0(f')q$ which equals $\phi_0(q)f' = 0$ modulo $\operatorname{Kos}(F)$. Define $L_1 := F/IF = F \otimes_R A$, which is a free A-module, and thus is graded. Set $d_2 \colon L_2 \to L_1$ to be the map induced by the inclusion $Q \hookrightarrow F$ and set $L_0 := \Omega_{R/B} \otimes_R A$. The R-module $\Omega_{R/B}$ is graded by setting $\deg(b\,dr) = \deg(b) + \deg(r)$ for all homogeneous $b \in B$ and $r \in R$. The tensor product is naturally graded (as both R- and A-module). Define $d_1 \colon L_1 \to L_0$ by the composition of $\overline{\phi_0} \colon F/IF \to I/I^2$ with the map $\partial \colon I/I^2 \to L_0$ taking $\overline{f} \mapsto df \otimes 1_A$. The map, d_1 is also graded of degree zero.

It is clear that $d_1 d_2 = 0$, because d_2 is induced by $\operatorname{Ker}(\phi_0) \hookrightarrow F$ and d_1 factors through $\overline{\phi_0}$. The *(truncated) cotangent complex of the homomorphism $B \to A$ with respect to M* is the (homotopy class of the) complex

$$L_\bullet \colon L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0.$$

Consider the M-dual complex

$$L^\bullet \colon L^0 \xrightarrow{d^1} L^1 \xrightarrow{d^2} L^2$$

of $L_\bullet$, where $L^i = \operatorname{Hom}_A(L_i, M)$ and d^i is the dual map of d_i. The *ith (upper) cotangent cohomology module* is defined to be the ith cohomology of the cotangent complex $L^\bullet$, and is denoted by $T^i(A/B, M)$ or $T^i_{A/B}(M)$. We also use the notation $T^i_{A/B}$, if $M = A$.

Tracking the gradings through this construction shows that $T^i_{A/B}(M)$ is a graded A-module (thus, also a graded B-module) for all i. Importantly, the T^i modules do not depend on the choices of R and F above, as is proved in [11, p. 20–1]. By construction, $T^i_{A/B}(-)$ is a covariant functor on $\mathbb{Z}$-graded A-modules. The associated sheaf to $T^i_{A/B}(M)$ on $X = \operatorname{Proj}(A)$ is denoted by $\mathscr{T}^i_{X/Y}(\mathscr{M})$, where $Y = \operatorname{Proj}(B)$ and $\mathscr{M} = \widetilde{M}$ is the associated sheaf on X. If $\mathscr{M} = \mathscr{O}_X$, then we write $\mathscr{T}^i_{X/Y}$, and if $Y = \operatorname{Spec}(\mathbb{F})$, we write $\mathscr{T}^i_X$. (The $\mathscr{T}^i$ sheaves exist for general morphisms $X \to Y$ and $\mathscr{O}_X$-modules by localization of the affine construction, as in [18, p. 16], but our presentation is needed for explicit computations.)

We now work out the cotangent complex explicitly in two important situations.

Key Example 2.1. Let $B = \mathbb{F}[x_0, \ldots, x_n]$ be the standard $\mathbb{Z}$-graded polynomial ring over a field $\mathbb{F}$, let $A = B/I$ be a graded quotient, and let M be a finitely generated graded A-module. In the construction of the cotangent complex of $B \twoheadrightarrow A$ with respect to M we can choose $R = B$. Let $f_1, \ldots, f_r$ be generators for I of degrees $\delta_1, \ldots, \delta_r$, so that $F = \bigoplus_{j=1}^r B(-\delta_j)$, and $Q = \operatorname{Ker}(\phi_0)$ is the kernel of the natural surjection $F \twoheadrightarrow I$. For computer calculations, it is useful to extend this to the second syzygies. Hilbert's syzygy theorem guarantees the existence of a finite free resolution of I (of length $\leq n+1$), which *Macaulay2* computes explicitly via the command `res`, so we exploit this resolution as part of the available data. Thus, there is an exact sequence

$$0 \longrightarrow F_{n+1} \xrightarrow{\phi_{n+1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} I \longrightarrow 0$$

with $F_0 = F$ as above, such that $Q = \operatorname{Ker}(\phi_0) = \operatorname{Im}(\phi_1) = F_1/\operatorname{Ker}(\phi_1)$ is presented as a quotient of a free B-module. The A-module L_2 is defined as $Q/\operatorname{Kos}(F_0)$, and since ϕ_1 surjects onto Q, lifting $\operatorname{Kos}(F_0)$ to $\operatorname{Kos}(F_0)' \subseteq F_1$ gives $L_2 = F_1/\operatorname{Kos}(F_0)'$. We observe that $L_1 = F_0/IF_0 = F_0 \otimes_B A$ is the free A-module $\bigoplus_{j=1}^r A(-\delta_j)$. The A-module L_0 is trivial, because $\Omega_{R/B} = \Omega_{B/B} = (0)$. Since d_2 is induced by the inclusion of $Q \subseteq F = F_0$, which is now represented by ϕ_1, the cotangent complex is

$$L^\bullet\colon 0 \xrightarrow{d^1} \operatorname{Hom}_A(\textstyle\bigoplus_{j=1}^r A(-\delta_j), M) \xrightarrow{d^2} \operatorname{Hom}_A(F_1/\operatorname{Kos}(F_0)', M)$$

where $d^2 = \overline{\phi}_1^t$ is the transpose of ϕ_1 considered as a matrix with entries in A. We find that

$$T^0_{A/B}(M) = (0), \qquad T^1_{A/B}(M) = \operatorname{Ker}(\overline{\phi}_1^t), \qquad T^2_{A/B}(M) = \operatorname{Coker}(\overline{\phi}_1^t).$$

Using $\operatorname{Ker}(d^2) = \{\phi : L_1 \to M \mid \operatorname{Ker}(\phi) \supseteq \operatorname{Im}(d_2)\}$, we find that $T^1_{A/B}(M) = \operatorname{Hom}_A(I/I^2, M)$. If $M = A$, then this implies that $T^1_{A/B} = N_{A/B} := \operatorname{Hom}_A(I/I^2, A)$ is the *normal module* of A over B.

Example 2.2. The cotangent cohomology modules of a plane curve $X \subseteq \mathbb{P}^2$ are easy to describe. Let $B = \mathbb{F}[x, y, z]$ and let $f \in B_\delta$ a homogeneous element defining X. The exact sequence $0 \to B(-\delta) \to I_X \to 0$ given by multiplication by f is a free B-resolution of the ideal $I_X = (f)$ of X. By the description above, the cotangent cohomology of $B \twoheadrightarrow A$ with respect to any M is $T^0_{A/B}(M) = T^2_{A/B}(M) = (0)$ and $T^1_{A/B}(M) = \operatorname{Hom}_A(A(-\delta), M)$. If X is a conic and $M = A$, then $T^1_{A/B} = A(2)$, so $(N_{A/B})_0 = (T^1_{A/B})_0 = A_2 \cong \mathbb{F}^5$, since $B_2 \cong \mathbb{F}^6$ and we quotient by the $\mathbb{F}$-span of f.

Key Example 2.3. Let us now compute the cotangent cohomology of $\mathbb{F} \to A$ with respect to M. Since $R = B$, we use a free resolution of I over B to compute the cotangent complex. As before, we have $L_2 = Q/\operatorname{Kos}(F_0) = F_1/\operatorname{Kos}(F_0)'$ and $L_1 = \bigoplus_{j=1}^r A(-\delta_j)$, but now L_0 is different. Indeed, $L_0 = \Omega_{R/\mathbb{F}} \otimes_R A = \Omega_{B/\mathbb{F}} \otimes_B A$ with $\Omega_{B/\mathbb{F}} = \bigoplus_{j=0}^n B \cdot dx_j$, which shows that L_0 is equal to $\bigoplus_{j=0}^n A \cdot dx_j = A(-1)^{n+1}$. Thus, $d_1\colon L_1 \to L_0$ is a map of free A-modules, $d_1\colon \bigoplus_{j=1}^r A(-\delta_j) \to A(-1)^{n+1}$. Since d_1 is induced by the composition of $F_0/IF_0 \to I/I^2$ with $\overline{f} \mapsto df \otimes 1_A$, we find that d_1 equals the transpose Jacobian $\overline{\operatorname{Jac}(\phi_0)}^t = \overline{\operatorname{Jac}(f_1, \ldots, f_r)}^t$

of the generators $f_1, \dots, f_r$, whose (k,ℓ)-entry is $\dfrac{\partial f_\ell}{\partial x_k}$ mod I. Hence, the M-dual of the cotangent complex is

$$L^\bullet \colon \operatorname{Hom}_A\big(A(-1)^{n+1}, M\big) \xrightarrow{\ d^1\ } \operatorname{Hom}_A\big(\textstyle\bigoplus_{j=1}^r A(-\delta_j), M\big) \xrightarrow{\ d^2\ } \operatorname{Hom}_A\big(F_1/\operatorname{Kos}(F_0)', M\big)$$

where $d^1 = \overline{\operatorname{Jac}(\phi_0)}$. This shows that

$$T^0_{A/\mathbb{F}}(M) = \operatorname{Ker}(\overline{\operatorname{Jac}(\phi_0)}), \quad T^1_{A/\mathbb{F}}(M) = T^1_{A/B}(M)/\operatorname{Im}(\overline{\operatorname{Jac}(\phi_0)}), \quad T^2_{A/\mathbb{F}}(M) = T^2_{A/B}(M).$$

In particular, if $M = A$, then the dual complex is

$$L^\bullet \colon A(1)^{n+1} \xrightarrow{\ \overline{\operatorname{Jac}(\phi_0)}\ } \textstyle\bigoplus_{j=1}^r A(\delta_j) \xrightarrow{\ \overline{\phi}_1^t\ } \operatorname{Hom}_A(F_1/\operatorname{Kos}(F_0)', A),$$

which gives a form of the cotangent complex amenable to explicit computer-aided computation of the cotangent cohomology modules. The module $T^1_{A/\mathbb{F}} = N_{A/B}/\operatorname{Im}(\overline{\operatorname{Jac}(\phi_0)})$ is the cokernel of the Jacobian map to $N_{A/B} = \operatorname{Ker}(\overline{\phi}_1^t)$, which enables us to compute $T^1_{A/\mathbb{F}}$ explicitly via the free resolution of I. If $A = B$, so that $I = (0)$ in the above construction, we must have $T^1_{B/\mathbb{F}}(M) = T^2_{B/\mathbb{F}}(M) = (0)$.

Example 2.4. To compute $T^1_{A/\mathbb{F}}$ for the conic curve $X = V(y^2 - xz)$, we compute the cokernel of the map $\overline{\operatorname{Jac}(f)} = \begin{bmatrix} -\overline{z} & 2\overline{y} & -\overline{x} \end{bmatrix} : A(1)^3 \to A(2)$. This is the quotient of $A(2)$ by the maximal ideal $\langle \overline{x}, \overline{y}, \overline{z} \rangle$. Thus, we obtain $T^1_{A/\mathbb{F}} = \mathbb{F}(2) \cong \mathbb{F}$. (If $\operatorname{char}(\mathbb{F}) = 2$, then $T^1_{A/\mathbb{F}} = A(2)/\langle \overline{x}, \overline{z} \rangle = (\mathbb{F}[y]/(y^2))\,(2) \cong \mathbb{F}^2$.)

It is also helpful to recognize the T^0 modules in a different guise (see [11, p. 23]).

Theorem 2.1. *For any homomorphism $B \to A$ of graded rings and any graded A-module M, we have $T^0_{A/B}(M) = \operatorname{Hom}_A(\Omega_{A/B}, M) = \operatorname{Der}_B(A, M)$. If $M = A$, then this implies that $T^0_{A/B} = T_{A/B}$ is the* tangent module *of A over B.*

Proof. Recall that $d_1 \colon L_1 \to L_0$ is defined by a composition $L_1 \twoheadrightarrow I/I^2 \to L_0$. The second map, ∂, appears in the *conormal sequence* of the quotient of B-algebras $R/I = A$ (i.e. the "second exact sequence" of [10, p. 173]). By surjectivity of the first map, we get a new exact sequence

$$\begin{array}{ccccccc} I/I^2 & \xrightarrow{\ \partial\ } & L_0 = \Omega_{R/B} \otimes_R A & \longrightarrow & \Omega_{A/B} & \longrightarrow & 0 \\ {\scriptstyle \overline{\phi}_0} \uparrow & \nearrow_{d_1} & & & & & \\ L_1 = F_0/IF_0 & & & & & & \end{array}$$

showing that $\Omega_{A/B} = \operatorname{Coker}(d_1)$. Taking dual modules with respect to M shows that $\operatorname{Hom}_A(\Omega_{A/B}, M) = \operatorname{Ker}(d^1) = T^0_{A/B}(M)$. The tangent module is $T_{A/B} = \operatorname{Hom}_A(\Omega_{A/B}, A)$ by definition. ♫

To relate various cotangent cohomology modules and enable computations, we have the following exact sequence.

Theorem 2.2. *Let $C \to B$ and $B \to A$ and be homomorphisms of graded rings, and M a graded A-module. There is a long exact sequence of cotangent cohomology modules*

$$\begin{array}{c} 0 \longrightarrow T^0_{A/B}(M) \longrightarrow T^0_{A/C}(M) \longrightarrow T^0_{B/C}(M) \\ \longrightarrow T^1_{A/B}(M) \longrightarrow T^1_{A/C}(M) \longrightarrow T^1_{B/C}(M) \\ \longrightarrow T^2_{A/B}(M) \longrightarrow T^2_{A/C}(M) \longrightarrow T^2_{B/C}(M). \end{array}$$

Sketch of proof. This is proved in the non-graded case in [11, p. 22–3] by judicious choices in the construction of the cotangent complex. Following the proof, all choices can be made to preserve gradings. ♫

For instance, if $C = \mathbb{F} \hookrightarrow B = \mathbb{F}[x_0, \dots, x_n]$ and $B \twoheadrightarrow A = B/I$, then we know that $T^0_{A/B}(M) = T^1_{B/\mathbb{F}}(M) = T^2_{B/\mathbb{F}}(M) = (0)$ from our previous calculations. This yields an exact sequence

$$0 \longrightarrow T^0_{A/\mathbb{F}}(M) \longrightarrow T^0_{B/\mathbb{F}}(M) \longrightarrow T^1_{A/B}(M) \longrightarrow T^1_{A/\mathbb{F}}(M) \longrightarrow 0$$

and an isomorphism $T^2_{A/B}(M) \cong T^2_{A/\mathbb{F}}(M)$. Setting $M = A$ in the exact sequence, we have

$$0 \longrightarrow T_{A/\mathbb{F}} \longrightarrow T_{B/\mathbb{F}} \longrightarrow N_{A/B} \longrightarrow T^1_{A/\mathbb{F}} \longrightarrow 0$$

giving a generalized *normal sequence* (dual to the conormal sequence) for arbitrary (possibly singular) subschemes of $\mathbb{P}^n$. In terms of the associated sheaves, we have

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{T}_{\mathbb{P}^n}|_X \longrightarrow \mathscr{N}_{X/\mathbb{P}^n} \longrightarrow \mathscr{T}^1_X \longrightarrow 0,$$

which shows that $\mathscr{T}^1_X$ is the cokernel of the sheaf morphism $\mathscr{T}_{\mathbb{P}^n}|_X \to \mathscr{N}_{X/\mathbb{P}^n}$. This is the definition of $\mathscr{T}^1_X$ given in [8, p. 99].

Example 2.5. Since the module of Kähler differentials of the coordinate ring of $\mathbb{P}^2$ equals $\Omega_{B/\mathbb{F}} = B(-1)^{n+1}$, our previous computations for the plane conic $X = V(y^2 - xz)$ fit together into the long exact sequence

$$0 \longrightarrow \operatorname{Ker}(\overline{\operatorname{Jac}(f)}) \longrightarrow B(1)^3 \otimes_B A = A(1)^3 \xrightarrow{\overline{\operatorname{Jac}(f)}} A(2) \longrightarrow \mathbb{F}(2) \longrightarrow 0$$

with $\overline{\operatorname{Jac}(f)} = \begin{bmatrix} -\overline{z} & 2\overline{y} & -\overline{x} \end{bmatrix}$ as above.

We now relate the cotangent cohomology modules with deformations. Let X be a locally finite type, separated $\mathbb{F}$-scheme. An infinitesimal family $\varphi\colon \mathcal{X} \to \operatorname{Spec}(S)$ for X is called *locally trivial* if there exists an open covering $\{U_i\}$ of X such that the corresponding open covering $\{\mathcal{X}_{U_i}\}$ of $\mathcal{X}$ satisfies the condition that every $\varphi|_{\mathcal{X}_{U_i}} : \mathcal{X}_{U_i} \to \operatorname{Spec}(S)$ is a trivial family for U_i. For any local artinian $\mathbb{F}$-algebra S with residue field $\mathbb{F}$, let $\mathsf{D}_X(S)$ denote the set of isomorphism classes of infinitesimal families for X over $\operatorname{Spec}(S)$, let $\mathsf{D}'_X(S) \subseteq \mathsf{D}_X(S)$ be the subset of locally trivial families, and, if X is projective, let $\mathsf{D}^{em}_X(S)$ denote the set of embedded families for $X \subseteq \mathbb{P}^n$ over $\operatorname{Spec}(S)$. Theorem 1.2 shows that first-order embedded families are given by $\mathsf{D}_X^{em}(\mathbb{F}[\varepsilon]) = T_{\mathcal{H}_p^n,[X]} = H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$, and we can also interpret the first-order non-embedded deformations of X cohomologically, by using the cotangent cohomology.

Theorem 2.3. *If X is a locally finite type, separated $\mathbb{F}$-scheme, then we have an isomorphism $\mathsf{D}'_X(\mathbb{F}[\varepsilon]) \cong H^1(X, \mathscr{T}_X)$, and an exact sequence*

$$0 \longrightarrow H^1(X, \mathscr{T}_X) \longrightarrow \mathsf{D}_X(\mathbb{F}[\varepsilon]) \longrightarrow H^0(X, \mathscr{T}_X^1) \longrightarrow H^2(X, \mathscr{T}_X),$$

which we refer to as the comparison sequence*, where $\mathscr{T}_X$ is the tangent sheaf of X over $\mathbb{F}$ and $\mathscr{T}_X^1$ is the first cotangent cohomology sheaf of X.*

Proof. This is proved in [18, p. 64–8], and in [11, p. 81] in a more general form. ♫

Example 2.6. Let $X = \mathrm{Proj}(A) \subseteq \mathbb{P}^2$ be a nonsingular curve of degree δ and genus g. The comparison sequence reduces to

$$0 \longrightarrow H^1(X, \mathscr{T}_X) \longrightarrow \mathsf{D}_X(\mathbb{F}[\varepsilon]) \longrightarrow H^0(X, \mathscr{T}_X^1) \longrightarrow 0.$$

In particular, the space of first-order families for X is at least as large as $H^1(X, \mathscr{T}_X)$. By Serre duality and Riemann–Roch we have $h^1(\mathscr{T}_X) \geq 3g-3$ (with equality if $g \geq 2$), giving a lower bound for $\dim_{\mathbb{F}}(\mathsf{D}_X(\mathbb{F}[\varepsilon]))$. (As an aside, we also see that if $g \geq 2$, then X has nontrivial locally trivial families, since $h^1(\mathscr{T}_X) > 0$.)

To get an upper bound, consider X as a Cartier divisor on $\mathbb{P}^2$. We have $\mathscr{N}_{X/\mathbb{P}^2} = \mathscr{O}_X(\delta)$ and $\deg(\mathscr{N}_{X/\mathbb{P}^2}) = X^2 = \delta^2$, where $X^2 = X \cdot X$ denotes self-intersection. The long exact sequence in cohomology associated to the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(\delta) \to \mathscr{O}_X(\delta) \to 0$ contains a surjection (of finite dimensional vector spaces) $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(\delta)) \twoheadrightarrow H^0(X, \mathscr{O}_X(\delta))$, which shows that $h^0(\mathscr{N}_{X/\mathbb{P}^2}) \leq h^0(\mathscr{O}_{\mathbb{P}^2}(\delta)) = \binom{2+\delta}{\delta}$.

Using $g = \frac{(\delta-1)(\delta-2)}{2}$, it follows that $3g-3 > \binom{2+\delta}{\delta}$ if and only if $\delta \geq 7$. Thus, these estimates show that if X is a plane curve of degree at least seven, then the space of first-order families for X is strictly larger than the space of first-order embedded families, that is, there exist non-embeddable deformations of the embedded curve $X \subseteq \mathbb{P}^2$.

Example 2.7. If $X = \mathrm{Spec}(A)$, then the comparison sequence reduces to an isomorphism $\mathsf{D}_X(\mathbb{F}[\varepsilon]) \cong H^0(X, \mathscr{T}_X^1) = T^1_{A/\mathbb{F}}$, showing that $T^1_{A/\mathbb{F}}$ parametrizes isomorphism classes of first-order families for X. If X is nonsingular, then $T^1_{A/\mathbb{F}} = (0)$ since the conormal sequence is exact, which implies that X has no nontrivial first-order deformations, i.e. X is *rigid*. Thus, if X is an arbitrary nonsingular scheme, then X is locally rigid, i.e. $\mathsf{D}'_X(\mathbb{F}[\varepsilon]) = \mathsf{D}_X(\mathbb{F}[\varepsilon])$.

Example 2.8. Let $X = \mathrm{Spec}(A) \subseteq \mathbb{A}^2$, where $A = \mathbb{F}[x,y]/\big(y^2 - x(x-1)(x-\lambda)\big)$, be an affine elliptic curve with $\lambda \neq 0, 1$. Since X is affine and nonsingular, it is rigid, having no nontrivial first-order deformations. However, the algebraic family $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$ for X over the base $D\big((t+\lambda)(t+\lambda-1)\big) \subseteq \mathbb{A}^1$ with variable t given by $\mathcal{A} = \mathbb{F}[x,y,t]/\big(y^2 - x(x-1)(x-(\lambda+t))\big)$ has fibres which are all mutually non-isomorphic elliptic curves, and is thus not trivial. Hence, X can be nontrivially deformed in an algebraic family, but not in any first-order family. This example appears in [18, p. 26].

The notion of *versality* links the deformation theory of $X \subseteq \mathbb{P}^n$ and the local geometry of the Hilbert scheme at $[X]$. The usual definition of versality is functorial, and rather abstract, however, in our primary case of interest it reduces to two pullback properties. If $X \subseteq \mathbb{P}^n$

is a closed subscheme, then $\mathsf{D}^{em}_X(-)$ naturally defines a covariant *functor of artinian rings* $(Art) \to (Sets)$ from the category (Art) of local artinian $\mathbb{F}$-algebras with residue field $\mathbb{F}$ to the category of sets. The functor D^{em}_X naturally extends to a functor $\widehat{\mathsf{D}^{em}_X}\colon (Com) \to (Sets)$, where (Com) is the category of local complete noetherian $\mathbb{F}$-algebras with residue field $\mathbb{F}$, defined by setting $\widehat{\mathsf{D}^{em}_X}(R) = \varprojlim D^{em}_X(R/\mathfrak{m}^{m+1})$, where $\mathfrak{m}$ is the maximal ideal of $R \in (Com)$ (using that $R/\mathfrak{m}^{m+1}$ is artinian for all $m \geq 0$). A *formal pair* for D^{em}_X is a pair $(R, \hat{\varphi})$ such that $\hat{\varphi} = \{\varphi^{(m)}\}_{m\geq 0} \in \widehat{\mathsf{D}^{em}_X}(R)$. Such $(R, \hat{\varphi})$ consists of a sequence of embedded families for X over the rings $R/\mathfrak{m}^{m+1}$, $m \geq 0$, giving cartesian squares as follows:

$$\left\{\begin{array}{ccccccccc} \cdots \hookleftarrow & \mathcal{X}^{(m)} & \hookleftarrow & \mathcal{X}^{(m-1)} & \hookleftarrow \cdots \hookleftarrow & \mathcal{X}^{(1)} & \hookleftarrow & \mathcal{X}^{(0)} = X \\ & \downarrow \varphi^{(m)} & & \downarrow \varphi^{(m-1)} & & \downarrow \varphi^{(1)} & & \downarrow \varphi^{(0)} \\ \cdots \hookleftarrow & \operatorname{Spec}(R/\mathfrak{m}^{m+1}) & \overset{\pi_m}{\hookleftarrow} & \operatorname{Spec}(R/\mathfrak{m}^{m}) & \hookleftarrow \cdots \hookleftarrow & \operatorname{Spec}(R/\mathfrak{m}^{2}) & \overset{\pi_1}{\hookleftarrow} & \operatorname{Spec}(\mathbb{F}) \end{array}\right\}$$

We think of $\hat{\varphi}$ as a possibly non-infinitesimal limit family $\widehat{\mathcal{X}} \to \operatorname{Spec}(R)$ over R.

A formal pair $(R, \hat{\varphi})$ for D^{em}_X is called *versal* if it satisfies the following two properties. First, given any embedded infinitesimal family $\psi\colon \mathcal{Y} \to \operatorname{Spec}(S)$ for X, we require that there exists a homomorphism $\mu\colon R \to S$ such that ψ equals the pullback $\varphi^{(m)}_S$ of $\varphi^{(m)}$ over $R/\mathfrak{m}^{m+1} \to S$ for all $m \gg 0$. Second, for every surjection $T \twoheadrightarrow S$ in (Art) and every ordered pair $\big(\mu\colon R \to S, \psi'\colon \mathcal{Y}' \to \operatorname{Spec}(T)\big)$ with $\psi' \in \mathsf{D}^{em}_X(T)$ such that the pullback families $\psi'_S = \varphi^{(m)}_S$ are equal for all $m \gg 0$, we require that there exists $\nu\colon R \to T$ such that $\mu = \pi \circ \nu$ and $\psi' = \varphi^{(m)}_T$ for $m \gg 0$. A versal pair is called *universal*, if given any ψ in the first property there exists a unique μ, and is called *miniversal* if this holds for each first-order ψ.

The first versality property roughly says that versal pairs have sufficiently many deformation parameters. The second property says that versal pullbacks always lift. For example, if $S = \mathbb{F}[u]/(u^r)$, $r \geq 1$, and $\psi\colon \mathcal{Y} \to \operatorname{Spec}(S)$, then the first property implies that there exists $\mu\colon R \to S$ such that $\psi = \varphi^{(m)}_S$ for $m \geq r-1$. Moreover, if $T = \mathbb{F}[u]/(u^{r+1})$, $T \twoheadrightarrow S$ is the canonical surjection, and $\psi'\colon \mathcal{Y}' \to \operatorname{Spec}(T)$, then the first property gives $\mu\colon R \to S$ such that $\psi'_S = \varphi^{(m)}_S$ for $m \geq r-1$, and by the second property there exists $\nu\colon R \to T$ lifting μ such that $\psi' = \varphi^{(m)}_T$ for $m \geq r$. A versal couple is "sufficiently universal" in the precise sense described by these properties.

Example 2.9. Let $X \subseteq \mathbb{P}^2 = \operatorname{Proj}(B)$ be the cuspidal cubic $V(y^2z - x^3)$. By the methods of the next section, the universal pair for D^{em}_X equals $(\mathbb{F}[[u_1, \ldots, u_9]], \hat{\varphi})$ where

$$\varphi^{(m)}\colon \operatorname{Proj}\left(\frac{B[[u_1, \ldots, u_9]]}{(F) + (u_1, \ldots, u_9)^{m+1}}\right) \to \operatorname{Spec}\left(\frac{\mathbb{F}[[u_1, \ldots, u_9]]}{(u_1, \ldots, u_9)^{m+1}}\right)$$

and $F = (y^2z - x^3 - u_1xz^2 - u_2z^3 - u_3x^2y - u_4x^2z - u_5xy^2 - u_6xyz - u_7y^3 - u_8y^2z - u_9yz^2)$. On the other hand, a miniversal pair for D^{em}_X equals $(\mathbb{F}[[u_1, u_2]], \hat{\psi})$ where

$$\psi^{(m)}\colon \operatorname{Proj}\left(\frac{B[[u_1, u_2]]}{(y^2z - x^3 - u_1xz^2 - u_2z^3) + (u_1, u_2)^{m+1}}\right) \to \operatorname{Spec}\left(\frac{\mathbb{F}[[u_1, u_2]]}{(u_1, u_2)^{m+1}}\right)$$

Other examples of versal pairs can be explicitly computed via the power series ansatz.

3. The Power Series Ansatz

Versal pairs can often be explicitly computed via the *power series ansatz*. This is a procedure for computing versal pairs for local moduli functors of artinian rings. We describe the sequence of steps used to compute a versal pair $(R, \hat{\varphi})$, $\hat{\varphi} = \{\varphi^{(m)}\}_{m \geq 0}$ for the functor D_X^{em}, where $X \subseteq \mathbb{P}^n$ is a closed subscheme with saturated graded ideal $I_X \subseteq B = \mathbb{F}[x_0, \ldots, x_n]$.

The ultimate goal of this calculation is to relate the versal pair to the geometry of the Hilbert scheme locally at $[X]$. If a comparison theorem applies, then the *versal base ring* R is isomorphic to the completion of the local ring $\mathscr{O}_{\mathcal{H}_p^n,[X]}$ of the Hilbert scheme at $[X]$. This approach proves fruitful in [17, p. 764]. We begin with a bare outline of the procedure, followed by a detailed description alongside a running example. We then reproduce the result of [17, p. 769–71] with our own *Macaulay2* implementation.

Overview. The main idea is to start with a general first-order family $\varphi^{(1)}$ for X over a deformation parameter ring $R = \mathbb{F}[[u_1, \ldots, u_k]]$, and successively build the versal deformation by lifting each $\varphi^{(m)}$ to $\varphi^{(m+1)}$. In the following, we use *mth-order* to refer to terms of, or up to, degree m in the variables $u_1, \ldots, u_k$. Each lift exists if and only if its associated obstruction (defined below) vanishes. We force this by iteratively adding obstruction equations to a base ideal for the ring R. If at some point no higher order nontrivial lifts are possible, the ansatz terminates, however, there is no guarantee that this occurs for any given example.

To commence the power series ansatz, we compute the first-order deformations of X. There is some choice as to whether one includes all deformations, or merely the nontrivial deformations. The difference determines whether we attempt to compute the universal pair, or a miniversal pair, respectively. Either way, we compute $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$ explicitly. To specialize to nontrivial families, one may compute $T^1_{A/\mathbb{F}}$, with $A = B/I_X$, via the cotangent complex. The defining ideal I_X is determined by the first map ϕ_0 of its free resolution, and we study families for X by perturbing the map ϕ_0 and its first syzygy map ϕ_1.

Denoting the computed basis vectors for $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$ by $\phi_0^1, \ldots, \phi_0^k$, the most general first-order family perturbing ϕ_0 is represented by $\Phi_0^{(1)} = \phi_0 + \sum_{i=1}^k u_i \phi_0^i$, introducing the free deformation parameters $u_1, \ldots, u_k$. Computing the first syzygies of this map gives $\Phi_1^{(1)}$, which perturbs ϕ_1 to first order and satisfies $\Phi_0^{(1)}\Phi_1^{(1)} \equiv 0 \mod (u_1, \ldots, u_k)^2$.

Having computed $\Phi_0^{(1)}$ and $\Phi_1^{(1)}$, we attempt to lift $\Phi_0^{(1)}$ to second order. In general, there is an obstruction in $T^2_{A/\mathbb{F}}$ to the existence of a lift, in our case expressed as a vector with second-order entries in $A \otimes_{\mathbb{F}} \mathbb{F}[[u_1, \ldots, u_k]]$ where $A = B/I_X$. Considering the deformation parameters as coefficients, we force the obstruction to vanish by setting all of its coefficients to zero. This modifies our versal base ring to $R = \mathbb{F}[[u_1, \ldots, u_k]]/J_0$ (up to second order), where J_0 is determined by the vanishing coefficients. Over these equations, the desired lift $\phi_0^{(2)}$ exists, and we then compute the perturbation $\phi_1^{(2)}$. This gives a solution to the equation

$$\Phi_0^{(2)}\Phi_1^{(2)} \equiv 0 \mod (u_1, \ldots, u_k)^3 + J_0.$$

Equivalently, letting Δ_0 denote a basis for $T^2_{A/\mathbb{F}}$ and letting Ω_0 denote the obstruction in $T^2_{A/\mathbb{F}}$, this solves the equation

$$\Phi_0^{(2)}\Phi_1^{(2)} + (\Delta_0\Omega_0)^t \equiv 0 \mod (u_1, \ldots, u_k)^3.$$

The abelian group $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$ acts on the set of all second-order lifts of $\Phi_0^{(1)}$ (we explain this in detail in the next section). After solving for second-order lifts, it is necessary to modify the choice of $\phi_0^{(2)}$ and $\phi_1^{(2)}$ to minimize the third-order terms of $\Phi_0^{(2)}\Phi_1^{(2)}$, which determine the next obstruction. We then check for a polynomial solution.

The rest of the algorithm proceeds similarly. We attempt to lift $\Phi_0^{(2)}$ to third order, and add additional, third-order terms to the obstruction equations if a nontrivial obstruction is encountered. We then compute the lifts $\phi_1^{(3)}$ of $\Phi_1^{(2)}$ and δ_1 of Δ_0 simultaneously, and check that the equation

$$\Phi_0^{(3)}\Phi_1^{(3)} + (\Delta_1\Omega_1)^t \equiv 0 \mod (u_1, \dots, u_k)^4$$

is satisfied. Following this, we minimize the fourth-order terms of $\Phi_0^{(3)}\Phi_1^{(3)}$ by modifying the third-order lifts, and check for a polynomial solution. Iterating, the process stops if we reach a polynomial solution over the deformation parameter ring $\mathbb{F}[[u_1, \dots, u_k]]$ to the equation

$$\Phi_0^{(\infty)}\Phi_1^{(\infty)} + (\Delta_\infty\Omega_\infty)^t = 0$$

with initial conditions $\Phi_0^{(0)} = \phi_0, \Phi_1^{(0)} = \phi_1$. If such a solution exists, then it determines the versal pair $(R, \hat{\varphi})$ such that $R = \mathbb{F}[[u_1, \dots, u_k]]/\operatorname{Im}(\Omega_\infty^t)$ and $\varphi^{(m)}: \mathcal{X}^{(m)} \to \operatorname{Spec}(R/\mathfrak{m}^{m+1})$ is the flat family for X defined by $\mathcal{X}^{(m)} = \operatorname{Proj}\left(B \otimes_{\mathbb{F}} (R/\mathfrak{m}^{m+1})/\operatorname{Im}(\Phi_0^{(m)})\right)$ with the canonical map to the base, for all $m \geq 0$.

A Running Example. Now we provide a more detailed description of the power series ansatz, and make the process explicit with a running example. Our approach follows a combination of the those of [19], [20], [14], and the source-code of the `versalDeformations.m2` package for *Macaulay2* (see [5]). After solving for the generic first-order family, the ansatz involves repeating the following five steps at each order of lifting: lifting the ideal, computing the obstruction, lifting the syzygies, minimizing the obstruction, and checking for a solution.

Step 1: Part One. Compute the most general first-order family for X.

The generic first-order family of deformations of X is given by computing an explicit basis of $T_{\mathcal{H}_p^n,[X]} = H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$. Let $\phi_0: F_0 \to B = \mathbb{F}[x_0, \dots, x_n]$ be the homomorphism sending the jth standard basis vector of $F_0 = \bigoplus_{j=1}^r B(-\delta_j)$ to the corresponding degree δ_j generator f_j of the graded ideal $I_X = (f_1, \dots, f_r)$ of X. The basis for $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$ is computed via the cotangent complex, resulting in r-vectors $\phi_0^1, \dots, \phi_0^k$ with entries in $A = B/I_X$. Lifting to B, we form the first-order perturbation $\Phi_0^{(1)} = \phi_0 + \phi_0^{(1)}$, where $\phi_0^{(1)} = \sum_{i=1}^k u_i\phi_0^i$.

Example 3.1. Let $A = \mathbb{F}[x, y, z]/(x, y, z)^2$ and $X = \operatorname{Spec}(A) \subseteq \mathbb{A}^3 \subseteq \mathbb{P}^3$. We compute the most general first-order family via the following *Macaulay2* commands.

```
Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : B = QQ[x,y,z];
i2 : I = (ideal(x, y, z))^2
                2             2         2
o2 = ideal (x  , x*y, x*z, y  , y*z, z  )
o2 : Ideal of B
```

We give the differentials the names used in the contruction of the cotangent complex.

```
i3 : (phi0, phi1) = (gens I, syz gens I)
o3 = (| x2 xy xz y2 yz z2 |, {2} | -y 0  -z 0  0  0  0  0  |)
                             {2} | x  -z 0  -y 0  -z 0  0  |
                             {2} | 0  y  x  0  0  0  0  -z |
                             {2} | 0  0  0  x  -z 0  0  0  |
                             {2} | 0  0  0  0  y  x  -z 0  |
                             {2} | 0  0  0  0  0  0  y  x  |
o3 : Sequence
```

In the Key Example 2.1, we show that $N_{A/B} = \text{Ker}\left(\overline{\phi}_1^t\right)$. The columns of the following matrix form a basis of $N_{A/B}$, which we lift to B, showing that $\dim_{\mathbb{F}}(N_{A/B}) = 18$.

```
i4 : A = B/I; basisN = lift(gens ker(A ** transpose phi1), B)
o5 = {-2} | z y x 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
     {-2} | 0 0 0 z y x 0 0 0 0 0 0 0 0 0 0 0 0 |
     {-2} | 0 0 0 0 0 0 z y x 0 0 0 0 0 0 0 0 0 |
     {-2} | 0 0 0 0 0 0 0 0 0 z y x 0 0 0 0 0 0 |
     {-2} | 0 0 0 0 0 0 0 0 0 0 0 0 z y x 0 0 0 |
     {-2} | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 z y x |
             6       18
o5 : Matrix B  <--- B
i6 : k = rank source basisN
o6 = 18
```

By Theorem 1.2, this shows that the point $[X] \in \mathcal{H}_4^3$ is singular, since $\mathcal{H}_4^3$ is irreducible of dimension 12, whose general point corresponds to 4 distinct points in $\mathbb{P}^3$. We define a new ring with deformation parameters $u_1, \ldots, u_k$ for $k = 18$.

```
i7 : B' = QQ[x,y,z][u_1..u_k];
i8 : (phi00, phi10) = (sub(phi0, B'), sub(phi1, B'));
```

We obtain $\Phi_0^{(1)}$, setting `phi01` to be its first-order part.

```
i9 : phi01 = vars B' * transpose sub(basisN, B'); transpose phi01
              1       6
o9 : Matrix B'  <--- B'
o10 = | u_1z+u_2y+u_3x    |
      | u_4z+u_5y+u_6x    |
      | u_7z+u_8y+u_9x    |
      | u_10z+u_11y+u_12x |
      | u_13z+u_14y+u_15x |
      | u_16z+u_17y+u_18x |
               6       1
o10 : Matrix B'  <--- B'
i11 : Phi01 = phi00 + phi01; transpose Phi01
               1       6
o11 : Matrix B'  <--- B'
o12 = | u_1z+u_2y+u_3x+x2    |
      | u_4z+u_5y+u_6x+xy    |
      | u_7z+u_8y+u_9x+xz    |
      | u_10z+u_11y+u_12x+y2 |
      | u_13z+u_14y+u_15x+yz |
      | u_16z+u_17y+u_18x+z2 |
               6       1
o12 : Matrix B'  <--- B'
```

Step 1: Part Two. Compute the first-order perturbation $\phi_1^{(1)}$ of ϕ_1.

We use a free resolution

$$0 \longrightarrow F_{n+1} \xrightarrow{\phi_{n+1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} I_X \longrightarrow 0$$

of the graded ideal I_X of X. The beginning of a free resolution of the ideal $I^{(1)} \subseteq B^{(1)}$ defined by $\Phi_0^{(1)}$ is $\cdots \to F_0^{(1)} \xrightarrow{\overline{\Phi}_0^{(1)}} I^{(1)} \to 0$ where $F_0^{(1)}$ and $B^{(1)}$ are the tensor products of F_0 and B with $\mathbb{F}[u_1, \ldots, u_k]/(u_1, \ldots, u_k)^2$. To compute $\Phi_1^{(1)}$, we apply the necessary condition $\Phi_0^{(1)}\Phi_1^{(1)} \equiv 0 \mod (u_1, \ldots, u_k)^2$, solving for an unknown $\phi_1^{(1)}$ satisfying

$$0 \equiv \Phi_0^{(1)}\Phi_1^{(1)} = \left(\phi_0 + \phi_0^{(1)}\right)\left(\phi_1 + \phi_1^{(1)}\right) \equiv \phi_0\phi_1 + \left(\phi_0\phi_1^{(1)} + \phi_0^{(1)}\phi_1\right) \mod (u_1, \ldots, u_k)^2.$$

Gathering first-order terms, this gives $\phi_0\phi_1^{(1)} = -\phi_0^{(1)}\phi_1$, where $\phi_0, \phi_0^{(1)}, \phi_1$ are known explicitly. Solving this equation for $\phi_1^{(1)}$ via matrix quotients gives the required perturbation of the first syzygies of ϕ_0. The *Macaulay2* matrix quotient command `f // g`, applied to matrices `f` and `g` with a common target, returns a quotient `q` such that `r = f - g*q` is the remainder of `f` modulo a Gröbner basis for the image of `g` (`r` is also directly obtained by `f % g`; see the documentation for `//` at [5]). By Theorem 1.2 and [18, p. 275], there exists such a $\phi_1^{(1)}$, that is, the remainder is zero. Continuing through the remaining resolution maps, we may lift the entire resolution to first-order in the same way.

Example 3.2. We continue with the fat point determined by the ideal $I_X = (x, y, z)^2$, solving for the first-order part of $\Phi_1^{(1)}$ using the matrix quotient command `//`.

```
i13 : phi11 = -phi01 * phi10 // phi00; Phi11 = phi10 + phi11
               6        8
o13 : Matrix B'  <--- B'
o14 = | -u_6-y     0        -u_9-z     -u_12       0            -u_15     ...
      | u_3-u_5+x -u_9-z   -u_8       u_6-u_11-y -u_15         -u_14-z   ...
      | -u_4       u_6+y    u_3-u_7+x -u_10       u_12          u_6-u_13 ...
      | u_2        -u_8     0          u_5+x      -u_14-z       0         ...
      | u_1        u_5-u_7 u_2        u_4         u_11-u_13+y u_5+x     ...
      | 0          u_4      u_1        0           u_10          u_4       ...
               6        8
o14 : Matrix B'  <--- B'
i15 : Phi01 * Phi11 % (ideal(u_1..u_k))^2 == 0
o15 = true
```

Hence, we have computed $\Phi_0^{(1)}$ and $\Phi_1^{(1)}$ such that $\Phi_0^{(1)}\Phi_1^{(1)} \equiv 0 \mod (u_1, \ldots, u_k)^2$.

Step 2: Part One. Compute the second-order lift $\phi_0^{(2)}$ of $\Phi_0^{(1)}$.

Following [20, p. 25] or [19, p. 135], we proceed by supposing that second-order terms $\phi_0^{(2)}$ and $\phi_1^{(2)}$ were found that lift the ideal and its syzygies to second order. If we let $\Phi_0^{(2)} = \Phi_0^{(1)} + \phi_0^{(2)}$ and $\Phi_1^{(2)} = \Phi_1^{(1)} + \phi_1^{(2)}$, then we would have

$$\Phi_0^{(2)}\Phi_1^{(2)} = \left(\Phi_0^{(1)} + \phi_0^{(2)}\right)\left(\Phi_1^{(1)} + \phi_1^{(2)}\right) = \Phi_0^{(1)}\Phi_1^{(1)} + \Phi_0^{(1)}\phi_1^{(2)} + \phi_0^{(2)}\Phi_1^{(1)} + \phi_0^{(2)}\phi_1^{(2)},$$

which would reduce to $\Phi_0^{(1)}\Phi_1^{(1)} + \phi_0\phi_1^{(2)} + \phi_0^{(2)}\phi_1 \equiv 0 \mod (u_1, \ldots, u_k)^3$. Considering this equation modulo the ideal I_X of X, we would have

$$\Phi_0^{(1)}\Phi_1^{(1)} + \phi_0^{(2)}\phi_1 \equiv 0 \mod (u_1, \ldots, u_k)^3 + I_X,$$

which would leave us with

$$\phi_0^{(2)}\phi_1 \equiv -\Phi_0^{(1)}\Phi_1^{(1)} \mod I_X$$

which makes sense, since $\Phi_0^{(1)}\Phi_0^{(1)}$ contains only second-order terms. In particular, this *lifting equation* puts a necessary condition on $\phi_0^{(2)}$. We attempt to solve the lifting equation for $\phi_0^{(2)}$ by matrix quotients.

Example 3.3. Continuing our example, we solve the lifting equation above.

```
i16 : I' = sub(I, B'); A' = B'/I';
o16 : Ideal of B'
i18 : phi02 = transpose(transpose(-Phi01 * Phi11 % I') // transpose phi10);
            transpose phi02
              1       6
o18 : Matrix B'  <--- B'
o19 = | u_3u_7-u_7^2-u_4u_8-u_1u_9+u_2u_13+u_1u_16         |
      | u_6u_7+u_5u_13-u_7u_13-u_4u_14-u_1u_15+u_4u_16     |
      | u_7u_9+u_8u_13-u_4u_17-u_1u_18                     |
      | u_7u_12+u_11u_13-u_13^2-u_10u_14-u_4u_15+u_10u_16 |
      | u_13u_14+u_7u_15-u_10u_17-u_4u_18                  |
      | -u_9^2-u_8u_15+u_9u_16+u_6u_17+u_3u_18-u_7u_18     |
              6       1
o19 : Matrix B'  <--- B'
i20 : Phi02 = Phi01 + phi02; transpose Phi02
              1       6
o20 : Matrix B'  <--- B'
o21 = | u_3u_7-u_7^2-u_4u_8-u_1u_9+u_2u_13+u_1u_16+u_1z+u_2y+u_3x+x2                |
      | u_6u_7+u_5u_13-u_7u_13-u_4u_14-u_1u_15+u_4u_16+u_4z+u_5y+u_6x+xy            |
      | u_7u_9+u_8u_13-u_4u_17-u_1u_18+u_7z+u_8y+u_9x+xz                            |
      | u_7u_12+u_11u_13-u_13^2-u_10u_14-u_4u_15+u_10u_16+u_10z+u_11y+u_12x+y2 |
      | u_13u_14+u_7u_15-u_10u_17-u_4u_18+u_13z+u_14y+u_15x+yz                      |
      | -u_9^2-u_8u_15+u_9u_16+u_6u_17+u_3u_18-u_7u_18+u_16z+u_17y+u_18x+z2         |
              6       1
o21 : Matrix B'  <--- B'
```

Step 2: Part Two. Compute the first obstruction to lifting.

After transposing the terms in the lifting equation, matrix division solves for $\phi_0^{(2)}$ in such a way that $-\Phi_0^{(1)}\Phi_1^{(1)} - \phi_0^{(2)}\phi_1$ is (the transpose of) a remainder upon factorization, defining the obstruction obs_0 to lifting the first-order perturbation $\Phi_0^{(1)}$ to second order. The obstruction obs_0 determines an element Ω_0 in $T^2_{A/\mathbb{F}}$, which, after computing an explicit basis of $T^2_{A/\mathbb{F}}$, is a vector with entries in $A \otimes_{\mathbb{F}} \mathbb{F}[[u_1, \ldots, u_k]]$. Treating the deformation parameters as coefficients, we force the obstruction to vanish by adding any nontrivial coefficients to an obstruction ideal J_0 and working over the new base $\mathbb{F}[[u_1, \ldots, u_k]]/J_0$.

Example 3.4. Continuing with the fat point of order four, we compute obs_0 and Ω_0.

```
i22 : obs0 = transpose(transpose(-Phi01 * Phi11 % I') % transpose phi10);
           transpose obs0
              1       8
o22 : Matrix B'  <--- B'
o23 = | -u_3u_4z-u_3u_5y+u_4u_5z+u_5^2y+u_1u_6z+u_2u_6y+u_5u_6x+u_3u_7y+...
      | -u_6u_8y+u_4u_9z+u_5u_9y-u_7u_9y+u_8u_10z+u_8u_11y+u_8u_12x-...
      | -u_3u_8y+u_5u_8y+u_6u_8x+u_7u_8y+u_2u_9y-u_8u_13x-u_2u_14y-...
```

```
       | -u_4u_6z-u_5u_6y-u_6^2x+u_6u_7y-u_5u_10z+u_7u_10z+u_8u_10y+...
       | -u_8u_12y-u_9u_12x+u_12u_14x+u_5u_15y+u_6u_15x-u_7u_15y-...
       | -u_6u_8y-u_6u_9x+u_8u_13y+u_9u_13x+u_6u_14x-u_13u_14x+u_2u_15y+...
       | u_9^2y-u_14^2y-u_9u_15x-u_14u_15x-u_9u_16y+u_14u_16y+u_15u_16x-...
       | -u_8u_9y-u_8u_14y+u_8u_16y+u_5u_17y-u_7u_17y+u_2u_18y

                8         1
o23 : Matrix B'  <--- B'
```

To compute Ω_0, we need a basis for $T^2_{A/\mathbb{F}}$. We use the description $T^2_{A/\mathbb{F}} = \text{Coker}\left(\overline{\phi}_1^t\right)$ obtained from the cotangent complex.

```
i24 : T2 = prune(Hom(image phi1/image koszul(2,phi0),A)/image(A ** transpose phi1));
i25 : for i from -5 to 5 list hilbertFunction(i, T2)
o25 = {0, 0, 0, 18, 0, 0, 0, 0, 0, 0, 0}
```

The module is concentrated in degree -2.

```
o25 : List
```

The columns of the following matrix give a basis of $T^2_{A/\mathbb{F}}$.

```
i26 : basisT2 = lift((gens image(T2.cache.pruningMap))*(gens image basis(-2,T2)),B)
o26 = {-3} | z 0 0 -y 0 0 0 0 0 0 x  0 0 0 0 0 0 0  |
      {-3} | 0 z x 0  0 0 0 0 0 0 -z 0 0 0 0 y 0 0  |
      {-3} | 0 0 0 0  y x 0 0 0 0 0  0 0 0 0 x 0 0  |
      {-3} | 0 0 0 0  0 0 z y x 0 -y 0 0 0 0 0 0 0  |
      {-3} | 0 0 0 0  0 0 0 0 0 x 0  0 0 y 0 0 0 0  |
      {-3} | 0 0 0 0  0 0 0 0 0 0 0  y x x 0 0 0 0  |
      {-3} | 0 0 0 0  0 0 0 0 0 0 0  0 0 0 x 0 0 -y |
      {-3} | 0 0 0 0  0 0 0 0 0 0 0  0 0 0 0 0 y 0  |

              8       18
o26 : Matrix B  <--- B
```

We obtain Ω_0 from obs_0 by finding its coefficients in terms of the given basis of $T^2_{A/\mathbb{F}}$.

```
i27 : Omega0 = transpose obs0 // sub(basisT2, B')
o27 = | -u_3u_4+u_4u_5+u_1u_6+u_4u_7-u_2u_10-u_1u_13
      | u_5u_6-u_6u_7+2u_4u_9+u_8u_10-u_2u_12-u_5u_13+u_7u_13-u_1u_15-...
      | u_8u_12-u_5u_15+u_7u_15-u_4u_18
      | u_3u_5-u_5^2-u_2u_6-u_3u_7+u_7^2+u_1u_9+u_2u_11-u_2u_13+u_1u_14-...
      | -u_3u_8+u_5u_8+u_7u_8+u_2u_9-u_2u_14-u_1u_17
      | 2u_6u_8-u_5u_9+u_7u_9-u_8u_11+u_5u_14-u_7u_14-u_2u_15+u_4u_17-...
      | -u_4u_6-u_5u_10+u_7u_10+u_4u_11+u_1u_12-u_4u_13
      | u_4u_9+u_8u_10-u_4u_14-u_1u_15
      | -u_6^2+u_9u_10+u_6u_11+u_3u_12-u_5u_12-u_7u_12-u_11u_13+u_13^2+...
      | -u_9u_12+u_12u_14+u_6u_15-u_11u_15+u_13u_15-u_10u_18
      | u_5u_6-u_6u_7+u_4u_9-u_2u_12-u_5u_13+u_7u_13+u_4u_14-u_4u_16
      | -u_6u_8+u_8u_13+u_2u_15-u_4u_17
      | -u_6u_9+u_8u_12+u_9u_13+u_6u_14-u_13u_14+u_3u_15-2u_5u_15+...
      | -u_8u_12+u_5u_15-u_7u_15+u_4u_18
      | -u_9u_15-u_14u_15+u_15u_16+u_12u_17+u_6u_18-u_13u_18
      | -u_6u_8+u_5u_9-u_7u_9+u_8u_11-u_8u_13-u_5u_14+u_7u_14+u_1u_18
      | -u_8u_9-u_8u_14+u_8u_16+u_5u_17-u_7u_17+u_2u_18
      | -u_9^2+u_14^2+u_9u_16-u_14u_16+u_6u_17-u_11u_17+u_13u_17+...

                18        1
o27 : Matrix B'   <--- B'
```

We also compute the obstruction ideal J_0.

```
i28 : J0 = ideal image transpose Omega0;
```

```
o28 = ideal (- u u  + u u  + u u  + u u  - u u   - u u  , u u  - u u  + 2u u  + ...
               3 4    4 5    1 6    4 7    2 10    1 13   5 6    6 7     4 9
o28 : Ideal of B'
```

Step 2: Part Three. Compute the second-order lift $\phi_1^{(2)}$ of the relations $\Phi_1^{(1)}$.

To solve for $\Phi_1^{(2)} = \Phi_1^{(1)} + \phi_1^{(2)}$, note that if such a lift were to exist, then we would have $\Phi_0^{(2)}\Phi_1^{(2)} + (\Delta_0\Omega_0)^t \equiv 0 \mod (u_1, \ldots, u_k)^3$, where Δ_0 is our basis of $T^2_{A/\mathbb{F}}$. Expanding and collecting second-order terms, this equals $\phi_0\phi_1^{(2)} + \phi_0^{(1)}\phi_1^{(1)} + \phi_0^{(2)}\phi_1 + (\Delta_0\Omega_0)^t = 0$, which gives the equation $\phi_0\phi_1^{(2)} = -(\phi_0^{(1)}\phi_1^{(1)} + \phi_0^{(2)}\phi_1 + (\Delta_0\Omega_0)^t)$, allowing us to solve for $\phi_1^{(2)}$ by matrix quotients. Again, we push this method through the entire resolution, if desired.

Example 3.5. Continuing our example, we compute the lifted relations.

```
i29 : Delta0 = sub(basisT2, B');

               8        18
o29 : Matrix B'  <--- B'
i30 : phi12 = -(phi02*phi10 + phi01*phi11 + transpose(Delta0*Omega0)) // phi00;
              Phi12 = Phi11 + phi12

               6        8
o30 : Matrix B'  <--- B'
o31 = | -u_6-y    0        -u_9-z    -u_12       0            -u_15    ...
      | u_3-u_5+x -u_9-z   -u_8      u_6-u_11-y -u_15         -u_14-z  ...
      | -u_4      u_6+y    u_3-u_7+x -u_10      u_12          u_6-u_13 ...
      | u_2       -u_8     0         u_5+x      -u_14-z       0        ...
      | u_1       u_5-u_7  u_2       u_4        u_11-u_13+y  u_5+x    ...
      | 0         u_4      u_1       0          u_10          u_4      ...

               6        8
o31 : Matrix B'  <--- B'
```

We check that this gives a solution modulo $(u_1, \ldots, u_k)^3$.

```
i32 : B'' = B'/J0; sub(Phi02 * Phi12, B'' / (ideal(u_1..u_k))^3) == 0
o33 = true
```

Equivalently, we check the following:

```
i34 : use B'; sub(Phi02*Phi12+transpose(Delta0*Omega0),B'/(ideal(u_1..u_k))^3) == 0
o35 = true
```

Let us make a brief interlude to better understand the structure of the set of lifts. If a lift $\phi_0^{(m)}$ exists for any $m \geq 1$, and $\phi_1^{(m)}$ is the corresponding lift on relations, then the set of all lifts of order m is a torsor under a $H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$-action [11, p. 47]. To see this, suppose that $\phi_0^{(m)}$ lifts $\Phi_0^{(m-1)}$, and for any vector $\psi_0 \in H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$, let ψ_1 satisfy $\phi_0\psi_1 + \psi_0\phi_1 = 0$. Let $\phi_0^{(m)} = \sum_{|\alpha|=m} u^\alpha \phi_0^\alpha$ denote the initial lift, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index of length k and $u^\alpha = u_i^{\alpha_1} \cdots u_k^{\alpha_k}$.

The action of ψ_0 on the lift $\phi_0^{(m)}$ is given by $(\psi_0, \phi_0^{(m)}) \mapsto \sum_{|\alpha|=m} u^\alpha(\phi_0^\alpha + \psi_0)$. This modified lift has modified relations $\sum_{|\alpha|=m} u^\alpha(\phi_1^\alpha + \psi_1)$, where $\phi_0^\alpha\phi_1 + \phi_0\phi_1^\alpha = 0$ for all α. To verify this, denote $\Phi_0^{(m)} = \Phi_0^{(m-1)} + \phi_0^{(m)}$ and $\Phi_1^{(m)} = \Phi_1^{(m-1)} + \phi_1^{(m)}$, so that

$$\Big(\Phi_0^{(m-1)} + \sum_{|\alpha|=m} u^\alpha(\phi_0^\alpha + \psi_0)\Big)\Big(\Phi_1^{(m-1)} + \sum_{|\alpha|=m} u^\alpha(\phi_1^\alpha + \psi_1)\Big)$$

equals $\left(\Phi_0^{(m)} + \sum_{|\alpha|=m} u^\alpha \psi_0\right)\left(\Phi_1^{(m)} + \sum_{|\alpha|=m} u^\alpha \psi_1\right)$, which reduces to

$$\Phi_0^{(m)}\Phi_1^{(m)} + \Big(\phi_0(\sum_{|\alpha|=m} u^\alpha \psi_1) + (\sum_{|\alpha|=m} u^\alpha \psi_0)\phi_1\Big) \mod (u_1, \ldots, u_k)^{m+1},$$

which is equivalent to $\Phi_0^{(m)}\Phi_1^{(m)}+\Big(\sum_{|\alpha|=m} u^\alpha(\phi_0\psi_1+\psi_0\phi_1)\Big) \equiv 0+0 = 0 \mod (u_1, \ldots, u_k)^{m+1}$. Thus, the $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$-action does not change the product $\Phi_0^{(m)}\Phi_1^{(m)}$ up to order m. This shows that, despite our definition, the obstruction obs_{m-2} is independent of the choice of lift $\phi_0^{(m)}$. However, the terms of order $m+1$ do change under the action by addition of $\sum_{|\alpha|=m} u^\alpha(\phi_0^{(1)}\psi_1 + \psi_0\phi_1^{(1)})$. Since the order $m+1$ terms of $\Phi_0^{(m)}\Phi_1^{(m)}$ determine $\phi_0^{(m+1)}$ and the obstruction obs_{m-1}, it is necessary to choose the order m lift that minimizes these terms.

Step 2: Part Four. Modify the second-order lifts to minimize the next obstruction.

After solving for the lifts $\phi_0^{(2)}$ and $\phi_1^{(2)}$, the product $-\Phi_0^{(2)}\Phi_1^{(2)}$ may have residual third-order terms. That is, there may exist nontrivial third-order terms that would vanish under a different choice of $\phi_0^{(2)}$. We find this lift and eliminate these terms, because they would otherwise appear as obstructions in the lifting equation for the next order, even though they are not true obstructions. To do this, we form a matrix from the vectors $\phi_0^{(1)}\psi_1 + \psi_0\phi_1^{(1)}$, where ψ_0 ranges over the basis of $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$, and eliminate residual terms via matrix quotients.

Example 3.6. Continuing with the fat point, we compute the residual terms modulo J_0+I_X.

```
i36 : A'' = B'/(J0 + I'); resTerm1 = (Phi02 * Phi12) ** A''
o37 = 0

                   1        8
o37 : Matrix A''  <--- A''
```

The residual terms are trivial, so no simplification is necessary.

Step 2: Part Five. Check for a polynomial solution.

The procedure terminates when we find a polynomial solution $\Phi_0^{(m)}\Phi_1^{(m)}+(\Delta_{m-2}\Omega_{m-2})^t = 0$ for some $m \geq 1$. After eliminating residual higher-order terms, we check whether this occurs.

Example 3.7. We check our example for a polynomial solution.

```
i38 : Phi02 * Phi12 + transpose(Delta0 * Omega0) == 0
o38 = false
```

We do not have a solution to $\Phi_0^{(2)}\Phi_1^{(2)} + (\Delta_0\Omega_0)^t \equiv 0$. However, note the following:

```
i39 : sub(Phi02 * Phi12, B'') == 0
o39 = true
```

This implies that there must be some terms missing from from our final matrix Δ_∞.

Step 3: Part One. Compute the third-order lift $\phi_0^{(3)}$ of $\Phi_0^{(2)}$.

Having solved to second order, obtaining $\Phi_0^{(2)} = \phi_0 + \phi_0^{(1)} + \phi_0^{(2)}$ and $\Phi_1^{(2)} = \phi_1 + \phi_1^{(1)} + \phi_1^{(2)}$ such that $\Phi_0^{(2)}\Phi_1^{(2)} \equiv 0 \mod (u_1, \ldots, u_k)^3$, to compute a third-order lift $\phi_0^{(3)}$ we derive the lifting equation

$$\phi_0^{(3)}\phi_1 \equiv -(\Phi_0^{(2)}\Phi_1^{(2)}) \mod (u_1, \ldots, u_k)^4 + I_X$$

in the same manner as for the previous lifting equation. By matrix quotients, we use this necessary condition to solve for $\phi_0^{(3)}$.

Example 3.8. Continuing our example, we compute the third-order lift $\phi_0^{(3)}$.

```
i40 : phi03 = lift(transpose((transpose(-Phi02*Phi12)**A'')//transpose(phi10**A'')),B');
                1       6
o40 : Matrix B'  <--- B'
i41 : Phi03 = Phi02 + phi03; Phi03 == Phi02
                1       6
o41 : Matrix B'  <--- B'
o42 = true
```

This lift is trivial, as expected from our polynomial solution check in the previous step.

Step 3: Part Two. Compute the second obstruction to lifting.

The obstruction $obs_1 = -\Phi_0^{(2)}\Phi_1^{(2)} - \phi_0^{(3)}\phi_1$ to lifting $\Phi_0^{(2)}$ to third order defines $\omega_1 \in T^2_{A/\mathbb{F}}$, and $\Omega_1 = \Omega_0 + \omega_1$ determines the ideal J_1 of extended equations of the base space.

Example 3.9. Continuing with the fat point of order four, we compute obs_1 and ω_1.

```
i43 : obs1 = lift(transpose((transpose(-Phi02*Phi12)**A'')%transpose(phi10**A'')),B');
                1       8
o43 : Matrix B'  <--- B'
i44 : omega1 = lift(transpose(obs1 ** A') // sub(basisT2, A'), B')
o44 = 0
                18       1
o44 : Matrix B'   <--- B'
i45 : Omega1 = Omega0 + omega1;
                18       1
o45 : Matrix B'   <--- B'
```

Step 3: Part Three. Compute the third-order lifts $\phi_1^{(3)}$ of $\Phi_1^{(2)}$ and δ_1 of Δ_0.

Example 3.10. Continuing our example, we compute $\phi_1^{(3)}$ and δ_1. We collect third-order terms of the product $\Phi_0^{(3)}\Phi_1^{(3)} + (\Delta_1\Omega_1)^t$, which vanishes to fourth order, and solve for the unknowns via matrix quotients.

```
i46 : simuLift1 = -(phi01*phi12+phi02*phi11+phi03*phi10+transpose(Delta0*omega1))
                  // (phi00|transpose Omega0)
o46 = | 0          0          0     0          0     0     0     0     |
      | 0          0          0     0          0     0     0     0     |
      | 0          0          0     0          0     0     0     0     |
      | 0          0          0     0          0     0     0     0     |
      | 0          0          0     0          0     0     0     0     |
      | 0          0          0     0          0     0     0     0     |
      | -u_14+u_16 0          -u_17 0          0     0     0     0     |
      | 0          0          0     0          0     0     0     0     |
      | -u_1       u_7        0     -u_4       -u_13 -u_7  0     0     |
      | -u_13      0          0     0          0     0     0     0     |
      | 0          0          u_13  0          0     0     0     0     |
      | 0          -u_13      0     0          0     0     0     0     |
      | 0          -u_17      0     -u_14+u_16 u_18  -u_17 0     0     |
      | 0          -u_14+u_16 0     u_13       u_15  0     0     u_17  |
      | 0          0          0     u_7        0     0     0     0     |
      | 0          0          0     u_1        u_7   0     0     0     |
```

```
      | u_7        0          0     -u_13      0      0     0     0     |
      | u_4        -u_13      -u_7  0          0      u_13  0     0     |
      | u_1        0          0     0          0      u_7   0     0     |
      | 0          0          0     0          0      0     0     0     |
      | 0          u_1        0     0          u_4    u_1   u_7   0     |
      | 0          0          0     0          0      0     0     0     |
      | 0          0          u_4   0          0      0     0     u_13  |
      | 0          -u_4       0     0          -u_10  -u_4  -u_13 0     |

               24        8
o46 : Matrix B'   <--- B'
i47 : phi13 = lift(simuLift1^(toList(0..(numgens target phi1)-1)),B');
              Phi13 = Phi12+phi13;
               6        8
o47 : Matrix B'  <--- B'
               6        8
o48 : Matrix B'  <--- B'
```

The bottom portion gives the lift δ_1.

```
i49 : delta1 = transpose lift(simuLift1^(
                 toList((numgens target phi1)..(numgens target phi1)+17)),B');
               8        18
o49 : Matrix B'  <--- B'
i50 : Delta1 = Delta0 + delta1
o50 = | -u_14+u_16+z 0 -u_1   -u_13-y 0      0     0            0            ...
      | 0            z u_7+x  0       0      -u_13 -u_17        -u_14+u_16   ...
      | -u_17        0 0      0       u_13+y x     0            0            ...
      | 0            0 -u_4   0       0      0     -u_14+u_16+z u_13+y       ...
      | 0            0 -u_13  0       0      0     u_18         u_15         ...
      | 0            0 -u_7   0       0      0     -u_17        0            ...
      | 0            0 0      0       0      0     0            0            ...
      | 0            0 0      0       0      0     0            u_17         ...

               8        18
o50 : Matrix B'  <--- B'
```

We verify that we have found a solution up to fourth order.

```
i51 : use B'; sub(Phi03*Phi13+transpose(Delta1*Omega1),B'/(ideal(u_1..u_k))^4) == 0
o52 = true
```

Step 3: Part Four. Modify the third-order lifts to minimize the next obstruction.

Example 3.11. We compute the residual terms.

```
i53 : resTerm2 = (Phi03 * Phi13) ** A''
o53 = 0
                1         8
o53 : Matrix A''  <--- A''
```

Step 3: Part Five. Check for a polynomial solution.

Example 3.12. We check the following:

```
i54 : Phi03 * Phi13 + transpose(Delta1 * Omega1) == 0
o54 = true
```

Hence, we have computed an explicit polynomial solution to $\Phi_0^{(\infty)}\Phi_1^{(\infty)} + (\Delta_\infty\Omega_\infty)^t = 0$ over $B' = B[[u_1, \ldots, u_{18}]]$. We have $\Phi_0^{(\infty)} = \Phi_0^{(2)}$, $\Phi_1^{(\infty)} = \Phi_1^{(1)}$, $\Delta_\infty = \Delta_1$, and $\Omega_\infty = \Omega_0$. The corresponding versal pair for D_X^{em} has base ring $R = \mathbb{F}[[u_1, \ldots, u_{18}]]/\operatorname{Im}(\Omega_0^t)$ and versal family $\hat{\varphi} = \{\varphi^{(m)}\}_{m\geq 0}$ where $\varphi^{(m)}\colon \operatorname{Proj}\left(B \otimes_{\mathbb{F}} (R/\mathfrak{m}^{m+1})/\operatorname{Im}\left(\Phi_0^{(2)}\right)\right) \to \operatorname{Spec}(R/\mathfrak{m}^{m+1})$ for all $m \geq 2$.

The Degenerate Twisted Cubic. Let $X \subseteq \mathbb{P}^3$ be the degenerate twisted cubic defined by the ideal $I_X = (xz, yz, z^2, x^3)$. We reproduce the result of [17, p. 769-70], computing the universal pair for D_X^{em} by implementing the power series ansatz in *Macaulay2*.

Step 1: Part One. Compute the most general first-order family for X.

```
Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : B = QQ[x,y,z,w];
i2 : I = monomialIdeal(x*z, y*z, z^2, x^3);
o2 : MonomialIdeal of B
i3 : (phi0, phi1) = ((res I).dd_1, (res I).dd_2)
o3 = (| xz yz z2 x3 |, {2} | -y -z 0  -x2 |)
                       {2} | x  0  -z 0   |
                       {2} | 0  x  y  0   |
                       {3} | 0  0  0  z   |
o3 : Sequence
```

We compute a basis for the degree zero piece of the normal module $N_{A/B}$.

```
i4 : A = B/I; N = prune ker(A ** transpose phi1);
     basisN = lift((gens image(N.cache.pruningMap))*(gens image basis(0,N)),B)
o6 = {-2} | 0   0  0   0   0   0   0   0   0   zw 0  x2 xy xw 0  0  |
     {-2} | 0   0  0   0   0   0   0   0   0   0  zw xy y2 yw 0  x2 |
     {-2} | 0   0  0   0   0   0   0   0   0   0  0  0  0  0  zw 0  |
     {-3} | zw2 y3 y2w yw2 x2y x2w xy2 xyw xw2 0  0  0  0  0  0  0  |
              4      16
o6 : Matrix B  <--- B
```

Though this basis is not identical to the basis of [17, p. 769-70], their spans are equal. Thus, $\dim_K(T_{\mathcal{H}^3_{3t+1},[X]}) = 16$, as is proved in [17, p. 766]. We use this basis for $\phi_0^{(1)}$.

```
i7 : k = rank source basisN; B' = B[u_1..u_k];
i9 : (phi00, phi10) = (sub(phi0, B'), sub(phi1, B'));
i10 : phi01 = vars B' * transpose sub(basisN, B'); transpose phi01
               1       4
o10 : Matrix B'  <--- B'
o11 = {0, -2} | u_10zw+u_12x2+u_13xy+u_14xw                                 |
      {0, -2} | u_11zw+u_12xy+u_13y2+u_14yw+u_16x2                          |
      {0, -2} | u_15zw                                                      |
      {0, -3} | u_1zw2+u_2y3+u_3y2w+u_4yw2+u_5x2y+u_6x2w+u_7xy2+u_8xyw+u_9xw2 |
               4       1
o11 : Matrix B'  <--- B'
i12 : Phi01 = phi00 + phi01; transpose Phi01
               1       4
o12 : Matrix B'  <--- B'
o13 = {0, -2} | u_10zw+u_12x2+u_13xy+u_14xw+xz                                 |
      {0, -2} | u_11zw+u_12xy+u_13y2+u_14yw+u_16x2+yz                          |
      {0, -2} | u_15zw+z2                                                      |
      {0, -3} | u_1zw2+u_2y3+u_3y2w+u_4yw2+u_5x2y+u_6x2w+u_7xy2+u_8xyw+u_9xw2+x3 |
```

```
                4       1
o13 : Matrix B'  <--- B'
```

Step 1: Part Two. Compute the first-order perturbation $\phi_1^{(1)}$ of ϕ_1.

```
i14 : phi11 = -phi01 * phi10 // phi00; Phi11 = phi10 + phi11
                4       4
o14 : Matrix B'  <--- B'
o15 = {0, 2} | -u_11w-y u_12x+u_13y+u_14w-u_15w-z u_16x                    ...
      {0, 2} | u_10w+x  0                         u_12x+u_13y+u_14w-u_15w-z ...
      {0, 2} | 0        u_10w+x                   u_11w+y                  ...
      {0, 3} | -u_16    0                         0                        ...
                4       4
o15 : Matrix B'  <--- B'
```

Hence, we have computed $\Phi_0^{(1)}$ and $\Phi_1^{(1)}$. We verify that this is a first-order solution.

```
i16 : Phi01 * Phi11 % (ideal(u_1..u_k))^2 == 0
o16 = true
```

Step 2: Part One. Compute the second-order lift $\phi_0^{(2)}$ of $\Phi_0^{(1)}$.

```
i17 : I' = sub(I, B'); A' = B'/I';
o17 : Ideal of B'
i19 : phi02 = lift(transpose(transpose((-Phi01*Phi11)**A')//transpose(sub(phi1,A'))),B');
              transpose phi02
                1       4
o19 : Matrix B'  <--- B'
o20 = {0, -2} | u_10u_13yw+u_10u_14w2-u_2u_16y2-u_3u_16yw-u_4u_16w2
      {0, -2} | -u_10u_12yw+u_11u_12xw+u_11u_13yw+u_11u_14w2+u_5u_16xy+u_6u_16xw+...
      {0, -2} | -u_12^2x2-2u_12u_13xy-u_13^2y2-2u_12u_14xw-2u_13u_14yw-u_14^2w2+...
      {0, -3} | u_9u_10w3+u_4u_11w3-u_1u_14w3+u_1u_15w3
                4       1
o20 : Matrix B'  <--- B'
i21 : Phi02 = Phi01 + phi02; transpose Phi02
                1       4
o21 : Matrix B'  <--- B'
o22 = {0, -2} | u_10u_13yw+u_10u_14w2-u_2u_16y2-u_3u_16yw-u_4u_16w2+u_10zw+...
      {0, -2} | -u_10u_12yw+u_11u_12xw+u_11u_13yw+u_11u_14w2+u_5u_16xy+...
      {0, -2} | -u_12^2x2-2u_12u_13xy-u_13^2y2-2u_12u_14xw-2u_13u_14yw-...
      {0, -3} | u_9u_10w3+u_4u_11w3-u_1u_14w3+u_1u_15w3+u_1zw2+u_2y3+u_3y2w+...
                4       1
o22 : Matrix B'  <--- B'
```

Step 2: Part Two. Compute the first obstruction to lifting.

```
i23 : obs0 = lift(transpose(transpose((-Phi01*Phi11)**A')%transpose(sub(phi1,A'))),B');
             transpose obs0
                1       4
o23 : Matrix B'  <--- B'
o24 = {0, -3} | u_1u_16zw2+u_9u_16xw2     |
      {0, -3} | -u_4u_16zw2               |
      {0, -3} | -2u_14u_16x2w+u_15u_16x2w |
      {0, -4} | 0                         |
                4       1
o24 : Matrix B'  <--- B'
```

We must compute a basis for the degree zero piece of $T^2_{A/K}$.

```
i25 : T2 = prune(Hom(image phi1/image koszul(2,phi0),A)/image((transpose phi1)**A));
```

The columns of the following matrix give a basis of $T^2_{A/K}$ in degree zero.

```
i26 : basisT2 = lift((gens image(T2.cache.pruningMap))*(gens image basis(0,T2)), B)
o26 = {-3} | zw2 xw2 0    0   |
      {-3} | 0   0   -zw2 0   |
      {-3} | 0   0   0    x2w |
      {-4} | 0   0   0    0   |
              4       4
o26 : Matrix B  <--- B
```

We obtain Ω_0 from obs_0 as follows.

```
i27 : Omega0 = transpose obs0 // sub(basisT2, B')
o27 = | u_1u_16            |
      | u_9u_16            |
      | u_4u_16            |
      | -2u_14u_16+u_15u_16 |
               4        1
o27 : Matrix B'  <--- B'
i28 : J0 = ideal image transpose Omega0
o28 = ideal (u u  , u u  , u u  , - 2u  u   + u  u  )
              1 16   9 16   4 16      14 16    15 16
o28 : Ideal of B'
```

Step 2: Part Three. Compute the second-order lift $\phi_1^{(2)}$ of the relations $\Phi_1^{(1)}$.

```
i29 : Delta0 = sub(basisT2, B');
               4        4
o29 : Matrix B'  <--- B'
i30 : phi12 = -(phi02*phi10 + phi01*phi11 + transpose(Delta0*Omega0)) // phi00;
              Phi12 = Phi11 + phi12
               4        4
o30 : Matrix B'  <--- B'
o31 = {0, 2} | -u_11w-y -u_10u_12w+u_12x+u_13y+u_14w-u_15w-z ...
      {0, 2} | u_10w+x  -u_2u_16y-u_3u_16w                   ...
      {0, 2} | 0        u_10w+x                              ...
      {0, 3} | -u_16    2u_13u_16                            ...
               4        4
o31 : Matrix B'  <--- B'
```

We check that this gives a solution modulo $(u_1,\ldots,u_k)^3$.

```
i32 : use B'; (Phi02*Phi12 + transpose(Delta0*Omega0))%(ideal(u_1..u_k))^3 == 0
o33 = true
i34 : B'' = B'/J0; sub(Phi02*Phi12, B''/(ideal(u_1..u_k))^3) == 0
o35 = true
```

We have $\Phi_0^{(2)}$, $\Phi_1^{(2)}$, Δ_0, and Ω_0 such that $\Phi_0^{(2)}\circ\Phi_1^{(2)} + (\Delta_0\circ\Omega_0)^t \equiv 0 \mod (u_1,\ldots,u_k)^3$.

Step 2: Part Four. Modify the second-order lifts to minimize the next obstruction.

```
i36 : A'' = B'/(J0 + I'); resTerm1 = (Phi02 * Phi12) ** A''; transpose resTerm1
                1         4
o37 : Matrix A''  <--- A''
```

```
o38 = {0, -3} | -u_10^2u_12yw2+u_10u_11u_12xw2+...
      {0, -3} | -u_10^2u_12u_13yw2-u_10^2u_12u_14w3+...
      {0, -3} | u_10^2u_12^2yw2-u_10u_11u_12^2xw2-...
      {0, -4} | -u_7u_10^2u_12y2w2-u_8u_10^2u_12yw3-...
                4         1
o38 : Matrix A''  <--- A''
```

To describe the $H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$-action on lifts, we use the following list, whose ith entry consists of the pair (ϕ_0^i, ϕ_1^i) of first-order perturbations corresponding to the ith basis vector of the normal module in degree zero.

```
i39 : actL = apply(gens B', i -> {contract(i,phi01), contract(i,phi11)});
```

We use the following matrix, whose ith column contains the modification to the lowest order residual terms of the product $\Phi_0^{(2)} \circ \Phi_1^{(2)}$ after acting on the lift by the ith basis vector.

```
i40 : actM = matrix{apply(actL, l -> transpose(l#0*phi11 + phi01*l#1))};
               4        16
o40 : Matrix B'  <--- B'
```

We compute coefficients.

```
i41 : simpResTerm1 = -transpose resTerm1 // ((transpose phi1|actM) ** A'');
                20         1
o41 : Matrix A''   <--- A''
i42 : coeffsM1 = lift(simpResTerm1^(toList((numgens target phi1)..
                 (numgens target phi1)+(numgens source actM)-1)),B')
               16        1
o42 : Matrix B'   <--- B'
```

This gives correction terms, which modify our lifts.

```
i43 : phi02corr = sum apply(#actL, i -> coeffsM1_(i,0)*(actL#i#0));
                  transpose phi02corr
               1        4
o43 : Matrix B'  <--- B'
o44 = {1, -2} | -1/2u_5u_11u_12xw-1/2u_5u_6u_16xw+1/2u_5u_10u_16xw+2u_10u_12xw+...
      {1, -2} | -1/2u_5u_11u_12yw-1/2u_5u_6u_16yw+1/2u_5u_10u_16yw+2u_10u_12yw+...
      {1, -2} | 0
      {1, -3} | 2u_6u_10xw2+u_8u_10yw2-3u_10^2xw2+2u_3u_11yw2+u_8u_11xw2
               4        1
o44 : Matrix B'  <--- B'
i45 : phi12corr = sum apply(#actL, i -> coeffsM1_(i,0)*(actL#i#1))
o45 = {0, 2} | 0 -1/2u_5u_11u_12w-1/2u_5u_6u_16w+1/2u_5u_10u_16w+... ...
      {0, 2} | 0 0                                                   ...
      {0, 2} | 0 0                                                   ...
      {0, 3} | 0 0                                                   ...
               4        4
o45 : Matrix B'  <--- B'
i46 : Phi02 = Phi02 + phi02corr; Phi12 = Phi12 + phi12corr;
               1        4
o46 : Matrix B'  <--- B'
               4        4
o47 : Matrix B'  <--- B'
i48 : phi02 = Phi02 - Phi01; phi12 = Phi12 - Phi11;
               1        4
o48 : Matrix B'  <--- B'
```

```
                4        4
o49 : Matrix B'  <--- B'
```

We verify that we are left with a solution modulo $(u_1, \ldots, u_k)^3$.

```
i50 : use B'; (Phi02*Phi12 + transpose(Delta0*Omega0))%(ideal(u_1..u_k))^3 == 0
o51 = true
```

Step 2: Part Five. Check for a polynomial solution.

```
i52 : Phi02 * Phi12 + transpose(Delta0 * Omega0) == 0
o52 = false
i53 : sub(Phi02 * Phi12, B'') == 0
o53 = false
```

The procedure continues with Step 3 in the same manner. It turns out that we already have the complete obstructions, so we summarize the rest of the computations. We obtain a polynomial solution to $\Phi_0^{(\infty)} \circ \Phi_1^{(\infty)} + (\Delta_\infty \circ \Omega_\infty)^t = 0$, where $\Phi_0^{(\infty)} = \Phi_0^{(6)}$, $\Phi_1^{(\infty)} = \Phi_1^{(4)}$, $\Delta_\infty = \Delta_7$, and $\Omega_\infty = \Omega_0$. The corresponding universal pair has base ring

$$R = \mathbb{F}[[u_1, \ldots, u_{16}]]/J_0 = \mathbb{F}[[u_1, \ldots, u_{16}]]/(u_{16}) \cap (u_1, u_4, u_9, -2u_{14} + u_{15}),$$

which is isomorphic to the base ring derived in [17, p. 769–70], and the versal family stabilizes at $n = 6$. This confirms that the (analytically) local geometry of the Hilbert scheme of twisted cubics $\mathcal{H}^3_{3t+1}$ at the point $[X]$ is a transverse intersection of two reduced 12-dimensional and 15-dimensional components, as in [17, p. 771].

4. Research Goals and the Future

We discuss where to go from here, in the short-term, medium-term, and long-term.

4.1. Short-term Goals.

New Examples. Having successfully implemented the power series ansatz in *Macaulay2* for a small number of examples, the most obvious thing to do next is to compute more examples! Specifically, the procedure gives good results for small numbers of points in $\mathbb{P}^3$, and for the classical case of twisted cubics, as we demonstrated above. These results were expected, since we know that $\mathcal{H}^3_p$ is irreducible for small numbers of points p, and the analysis of $\mathcal{H}^3_{3t+1}$ is done by hand in [17]. But what if we try to push the method of the twisted cubics paper through in more complex situations?

Specifically, we interpret the analysis done for the twisted cubics as follows: there are limited possibilities for which subschemes of $\mathbb{P}^3$ have Hilbert polynomial $p(t) = 3t + 1$, and studying these tells us the number and dimensions of components of $\mathcal{H}^3_p$; the scheme structure of $\mathcal{H}^3_p$ is studied by computations of Zariski tangent spaces of sufficiently general points; finally, the *comparison theorem* [17, p. 764] allows the study of the intersection of the components of $\mathcal{H}^3_p$ via the power series ansatz applied to a sufficiently general point of the intersection. Thus, the local geometry of the Hilbert scheme computed by the ansatz captures much of the global geometry. We will analyze Hilbert schemes of other objects similarly, using the *Macaulay2* implementation of the power series ansatz to augment our ability to analyze the local geometry. If the points we analyze are sufficiently general, in whatever context, this could lead to a global analysis.

Concretely, we will begin with the Hilbert scheme $\mathcal{H}^4_{4t+1}$ of rational quartic curves in $\mathbb{P}^4$ (or more generally $\mathcal{H}^n_{nt+1}$ for $n \geq 4$), and the Hilbert scheme of rational quartic surfaces in $\mathbb{P}^5$. In another vein, it may be possible to directly analyze $\mathcal{H}^3_p$, where $p \geq 11$ is a small number of points, via the power series ansatz at a well-chosen collection of p points.

Improvements to the Algorithm. The power series ansatz begins with a computation of a basis of the normal module to form the most general first-order family, represented by $\Phi_0^{(1)}$. For isolated singularities, we can restrict to the most general *nontrivial* family by computing instead a basis of $T^1_{A/K}$ via the cotangent complex. This reduces the number of deformation parameters, which speeds up the *Macaulay2* sessions. In fact, this is already implemented in the `versalDeformations.m2` package of [5]. For projective schemes, however, the analogous reduction is not currently implemented, since the relationship between nontrivial deformations and $\mathscr{T}^1_X$ is governed by the comparison sequence, and is more complicated.

But, for the degenerate twisted cubic above, computing the degree zero piece of $T^1_{A/K}$ does give a basis for the 10-dimensional space of nontrivial first-order perturbations of ϕ_0 matching that in [17, p. 769-70]. Moreover, the ansatz applied to this 10-dimensional space terminates faster than using the full 16-dimensional space of first-order families. Since our goal is to probe more complex cases, at the boundary of our computational ability, reducing the complexity of the ansatz for prjective schemes by using $T^1_{A/K}$ rather than $N_{A/B}$ is an important step.

4.2. **Medium-term Goals.**

Multigraded Hilbert Schemes. In [7], Haiman and Sturmfels prove the existence of quasiprojective *multigraded Hilbert schemes*, which parametrize ideals in multigraded rings with a specified generalized Hilbert function, and generalize Hilbert schemes and other related constructions. The comparison theorem of [17] relates the local geometry of Hilbert schemes to the local geometry of multigraded Hilbert schemes in specific instances, but we do not know whether this comparison theorem suffices in other cases of interest. Thus, we would like to gain a better understanding of the multigraded case, and how to use the geometry of multigraded Hilbert schemes to study the questions we encounter about Hilbert schemes. On the other hand, the geometry of multigraded Hilbert schemes is another possible avenue for research.

4.3. **Long-term Goals.**

Straddling the Unknown. The underlying idea for this project is that there is interesting geometry to be discovered at the border between well-behaved Hilbert schemes, and nasty ones. *Murphy's Law* [21], [8], mentioned in the first section, tells us that general Hilbert schemes have arbitrarily "bad" geometry. We have seen some hints of this in the first section. But Hilbert schemes of small numbers of points in small projective spaces are irreducible and smooth (see [4], [2]), and, as we have seen, the Hilbert scheme of twisted cubics has a rather simple and beautiful geometry that we can explcitly analyze via deformation theory and the power series ansatz. Our hope is that Hilbert schemes of other low-degree, low-dimensional objects have similarly nice geometry, which can be analyzed via computational methods. This would result in a better understanding of the "geography" of Hilbert schemes, as we would be pushing the boundaries of what has been discovered, and mapping out some of the well-behaved examples that will appear before all hell breaks loose.

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