HILBERT SCHEMES WITH TWO BOREL-FIXED POINTS IN ARBITRARY CHARACTERISTIC

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Abstract. We extend the recent classification of Hilbert schemes with two Borel-fixed points to arbitrary characteristic. We accomplish this by synthesizing Reeves' algorithm for generating strongly stable ideals with the basic properties of Borel-fixed ideals and the author's previous work classifying Hilbert schemes with unique Borel-fixed points.

1. Introduction

Hilbert schemes parametrizing closed subschemes with a fixed Hilbert polynomial in projective space are fundamental moduli spaces. In [Sta17], we provide a rough guide to aid in predicting the complexity of the geometry of Hilbert schemes. In particular, we classify Hilbert schemes with unique Borel-fixed points, over an algebraically closed or characteristic 0 field, and show that these Hilbert schemes are smooth and irreducible (they are also rational). Recently, Ramkumar [Ram19] has built on this, classifying Hilbert schemes with two Borel-fixed points and analyzing their local geometry, when the base field has characteristic 0. Here we show that the classification holds (with one very minor adjustment) over arbitrary algebraically closed fields.

Let $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^n_{\mathbb{K}}$ with Hilbert polynomial p, where $\mathbb{K}$ is an algebraically closed field. Macaulay classified Hilbert polynomials of homogeneous ideals in [Mac27]. Any such admissible Hilbert polynomial $\mathsf{p}(t)$ has a unique combinatorial expression of the form $\sum_{j=1}^{r} \binom{t+b_j-j+1}{b_j}$, for integers $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$, with $d := \deg \mathsf{p} = b_1$. Our main result is the following theorem.

Theorem 1.1. *Let $n \geq d+2$. The Hilbert scheme $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ has exactly two Borel-fixed ideals if and only if one of the following conditions holds:*

- *(i)* *(a)* $b_1 = 0$ *and* $r = 3$,
  - *(a')* $b_1 = 0$ *and* $r = 4$, *if* $n = 2$ *and* $\mathbb{K}$ *does not have characteristic* 2,
  - *(b)* $b_1 = \cdots = b_{r-1} = 1$ *and* $b_r = 0$, *for* $r-1 \neq 1, 3$,
  - *(c)* $b_1 = \cdots = b_{r-1} \geq 2$ *and* $b_r = 0$, *for* $r-1 \neq 1$,
- *(ii)* *(a)* $b_1 > b_2 = 0$ *and* $r = 3$,
  - *(b)* $b_1 = \cdots = b_{r-2} > b_{r-1} = 1$ *and* $b_r = 0$, *for* $r-2 \neq 2$, *and*
  - *(c)* $b_1 = \cdots = b_{r-2} > b_{r-1} \geq 2$ *and* $b_r = 0$.

A point on $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ is Borel-fixed if its (saturated) ideal is fixed by the linear action of the Borel group of upper-triangular matrices in $\mathrm{GL}_{n+1}(\mathbb{K})$ on the polynomial ring $\mathbb{K}[x_0, x_1, \ldots, x_n]$. Borel-fixed ideals play an important geometric role, because they function as markers for interesting local geometry on Hilbert schemes. Many fundamental properties of Hilbert schemes have been understood by passing from a homogeneous ideal to its generic initial ideal, which is Borel-fixed [BS87]. Hence, to analyze a chosen Hilbert scheme, it is beneficial to understand all of its Borel-fixed points. In characteristic 0, Borel-fixed ideals are strongly stable. The combinatorial criterion defining strongly stable ideals is robust enough that many of their properties are well understood. For example, if I is strongly stable, then the minimal free resolution of I is known [EK90], the regularity of I is the largest of the degrees of its minimal monomial generators [BS87], and the saturation of I is generated by Reeves' algorithm [Ree92]. This makes studying Borel-fixed ideals much easier than in general.

Strongly stable ideals, including lexicographic ideals, are always Borel-fixed—their definition is derived by describing Borel-fixed ideals in characteristic 0. However, in positive characteristic, Borel-fixed ideals generally satisfy a more difficult combinatorial condition and are less well understood (although some progress has been made [Par94], [HP01], [Sin07]). Minimal free resolutions and regularity are only understood in limited cases, and no algorithm to generate Borel-fixed ideals is available. Theorem 1.1 provides a class of Hilbert schemes with the feature that their Borel-fixed points are all strongly stable (as does the classification in [Sta17]). To obtain this classification, we rely on Pardue's description of the elementary properties of Borel-fixed ideals over infinite fields [Par94], in conjunction with [Sta17]. Despite the absence of Reeves' algorithm and the theory of expansions for Borel-fixed ideals in positive characteristic, certain recursive aspects of the algorithm still function to generate Borel-fixed ideals in the examples we consider. In particular, once we understand the strongly stable points on our Hilbert schemes (and "nearby" ones), the weaker exchange property of Borel-fixed ideals is sufficient to complete our classification.

Note that when $\mathbb{K}$ has characteristic 0, the local deformation theory at these Borel-fixed ideals is worked out in [Ram19].

Conventions. Throughout, $\mathbb{K}$ is an algebraically closed field, $\mathbb{N}$ is the nonnegative integers, $S := \mathbb{K}[x_0, x_1, \dots, x_n]$ is the standard $\mathbb{Z}$-graded polynomial ring, and $\mathfrak{m}_k := \langle x_0, x_1, \dots, x_k \rangle$ is its homogeneous ideal, for $0 \leq k \leq n$. The Hilbert function and polynomial of the quotient S/I by a homogeneous ideal I are denoted h_I and p_I, respectively.

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2. Admissible Hilbert Polynomials

Let $S := \mathbb{K}[x_0, x_1, \dots, x_n]$ denote the homogeneous coordinate ring of $\mathbb{P}^n_{\mathbb{K}}$ over an algebraically closed field $\mathbb{K}$. Let M be a finitely generated graded S-module. The ***Hilbert function*** $\mathsf{h}_M \colon \mathbb{Z} \to \mathbb{Z}$ of M is defined by $\mathsf{h}_M(i) := \dim_{\mathbb{K}} M_i$ for all $i \in \mathbb{Z}$. Every such M has a ***Hilbert polynomial*** $\mathsf{p}_M(t) \in \mathbb{Q}[t]$ such that $\mathsf{h}_M(i) = \mathsf{p}_M(i)$ for $i \gg 0$; see [BH93, Theorem 4.1.3]. For a homogeneous ideal $I \subset S$, we denote by h_I and p_I the Hilbert function and Hilbert polynomial of the *quotient module* S/I, respectively.

Notation 1. For integers j, k, set $\binom{j}{k} = \frac{j!}{k!(j-k)!}$ if $j \geq k \geq 0$ and $\binom{j}{k} = 0$ otherwise. For a variable t and $a, b \in \mathbb{Z}$, define $\binom{t+a}{b} = \frac{(t+a)(t+a-1)\cdots(t+a-b+1)}{b!} \in \mathbb{Q}[t]$ if $b \geq 0$, and $\binom{t+a}{b} = 0$ otherwise.

A polynomial is an ***admissible Hilbert polynomial*** if it is the Hilbert polynomial of a closed subscheme in some $\mathbb{P}^n$. Admissible Hilbert polynomials correspond to nonempty Hilbert schemes. We use the well-known classification discovered by Macaulay.

Proposition 2.1. *The following conditions are equivalent:*

(i) The polynomial $\mathsf{p}(t) \in \mathbb{Q}[t]$ *is a nonzero admissible Hilbert polynomial.*
(ii) There exist integers $e_0 \geq e_1 \geq \cdots \geq e_d > 0$ *such that* $\mathsf{p}(t) = \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$.
(iii) There exist integers $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$ *such that* $\mathsf{p}(t) = \sum_{j=1}^{r} \binom{t+b_j-j+1}{b_j}$.

Moreover, these correspondences are bijective.

Proof.

(i) $\Leftrightarrow$ (ii) This is proved in [Mac27]; see also [Har66, Corollary 3.3 and Corollary 5.7].
(i) $\Leftrightarrow$ (iii) This follows from [Got78, Erinnerung 2.4]; see also [BH93, Exercise 4.2.17].

Uniqueness of the sequences of integers also follows. $\square$

We always work with nonzero admissible Hilbert polynomials. The ***Macaulay expression*** of an admissible Hilbert polynomial p is its expression $\mathsf{p}(t) = \sum_{i=0}^{d} \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$, where $e_0 \geq e_1 \geq \cdots \geq e_d > 0$. The ***Gotzmann expression*** of p is its expression $\mathsf{p}(t) = \sum_{j=1}^{r} \binom{t+b_j-j+1}{b_j}$, for $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$. From these, we can read off the degree $d = b_1$, the leading coefficient $e_d/d!$, and the ***Gotzmann number*** r of p; the latter bounds the Castelnuovo–Mumford regularity of saturated ideals with Hilbert polynomial p. In particular, such ideals are generated in degrees up to r; see [IK99, p. 300-301].

If $\mathsf{p}(t)$ has Macaulay and Gotzmann expressions as above, then $r = e_0$ and the (nonnegative) partition $(b_1, b_2, \ldots, b_r)$ is conjugate to the partition $(e_1, e_2, \ldots, e_d)$; see [Sta17, Lemma 2.4]. In particular, the partition $(b_1, b_2, \ldots, b_r)$ has $e_i - e_{i+1}$ parts equal to i, for all $0 \leq i \leq d$. We refer to $(e_0, e_1, \ldots, e_d)$ as the ***Macaulay partition*** and $(b_1, b_2, \ldots, b_r)$ as the ***Gotzmann partition*** of p. From here on we use Gotzmann expressions of admissible Hilbert polynomials.

Let p have Gotzmann partition $(b_1, b_2, \ldots, b_r)$. There are two fundamental binary relations on admissible Hilbert polynomials. The first "lifts" or "integrates" p to the polynomial $\Phi(\mathsf{p})$ with partition $(b_1 + 1, b_2 + 1, \ldots, b_r + 1)$. The second "adds one," taking p to $A(\mathsf{p}) := 1 + \mathsf{p}$, which has partition $(b_1, b_2, \ldots, b_r, 0)$. These roughly correspond to the geometric operations of forming a cone over our projective scheme and forming the disjoint union with a reduced point, respectively. Both $\Phi(\mathsf{p})$ and $A(\mathsf{p})$ are admissible by Proposition 2.1.

3. A Geography of Hilbert Schemes

In [Sta17], we observe that the set of nonempty Hilbert schemes of the type $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ forms the vertices of a collection of infinite full binary trees.

Proposition/Definition 3.1. *For each positive codimension c, let $\mathcal{H}_c$ be the graph whose vertices are all nonempty Hilbert schemes* $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ *parametrizing codimension c subschemes and whose edges are all pairs* $\big(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n), \mathrm{Hilb}^{A(\mathsf{p})}(\mathbb{P}^n)\big)$ *and* $\big(\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n), \mathrm{Hilb}^{\Phi(\mathsf{p})}(\mathbb{P}^{n+1})\big)$, *where* p *is an admissible Hilbert polynomial and* $n = c + \deg \mathsf{p}$. *Then $\mathcal{H}_c$ is an infinite full binary tree with root* $\mathrm{Hilb}^1(\mathbb{P}^c) = \mathbb{P}^c$.

Proof. See [Sta17, Theorem 2.14]. $\square$

This framework gives a rough chart of the geography of Hilbert schemes, as it highlights the fact that certain properties of Hilbert schemes hold in a predictable manner. For instance, this leads to the following classification of Hilbert schemes with unique Borel-fixed points.

Theorem 3.2. *Let p be an admissible Hilbert polynomial and $c = n - \deg \mathsf{p}$. The lexicographic point is the unique Borel-fixed point on $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ if and only if one of the following holds:*

(i) $b_r > 0$,
(ii) $c \geq 2$ *and* $r \leq 2$,
(iii) $c = 1$ *and* $b_1 = b_r$, *or*
(iv) $c = 1$ *and* $r - s \leq 2$, *where* $b_1 = b_2 = \cdots = b_s > b_{s+1} \geq \cdots \geq b_r$.

Proof. See [Sta17, Theorem 1.1]. $\square$

The ***lexicographic point*** $[L_n^{\mathsf{p}}] \in \mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ is the point defined by the saturated lex-segment ideal $L_n^{\mathsf{p}} \subseteq S$ with Hilbert polynomial p. The lexicographic point is nonsingular and lies on a unique irreducible component $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ called the ***lexicographic*** or ***Reeves–Stillman*** component [RS97]. It is described in the next section.

4. Lexicographic Ideals

Lex-segment, or lexicographic, ideals are monomial ideals whose homogeneous pieces are spanned by maximal monomials in lexicographic order. They are Borel-fixed, and in many respects are the most important monomial ideals.

For any $u = (u_0, u_1, \dots, u_n) \in \mathbb{N}^{n+1}$, let $x^u = x_0^{u_0} x_1^{u_1} \cdots x_n^{u_n}$. The ***lexicographic ordering*** is the relation $\succ$ on the monomials in S defined by $x^u \succ x^v$ if the first nonzero coordinate of $u - v \in \mathbb{Z}^{n+1}$ is positive, where $u, v \in \mathbb{N}^{n+1}$.

Example 4.1. We have $x_0 \succ x_1 \succ \cdots \succ x_n$ in lexicographic order on $S = \mathbb{K}[x_0, x_1, \dots, x_n]$. If $n \geq 2$, then $x_0 x_2^2 \succ x_1^4 \succ x_1^3$.

For a homogeneous ideal $I \subseteq S$, lexicographic order gives rise to two associated monomial ideals. First, the ***lex-segment ideal*** for h_I is the monomial ideal $L_n^{\mathsf{h}_I} \subseteq S$ whose ith graded piece is spanned by the $\dim_{\mathbb{K}} I_i = \mathsf{h}_S(i) - \mathsf{h}_I(i)$ largest monomials in S_i, for all $i \in \mathbb{Z}$. The equality $\mathsf{h}_I = \mathsf{h}_{L_n^{\mathsf{h}_I}}$ holds by definition and $L_n^{\mathsf{h}_I}$ is a homogeneous ideal of S; see [Mac27, § II], [MS05, Proposition 2.21]. More importantly, the ***(saturated) lexicographic ideal*** for p_I is the lex-segment ideal $L_n^{\mathsf{p}_I} := \left(L_n^{\mathsf{h}_I} : \mathfrak{m}^\infty\right)$, where $\mathfrak{m} := \langle x_0, x_1, \dots, x_n \rangle \subset S$ is the irrelevant ideal. Saturation with respect to $\mathfrak{m}$ does not affect the Hilbert function in large degrees, so $L_n^{\mathsf{p}_I}$ also has Hilbert polynomial p_I.

Given a finite sequence of nonnegative integers $a_0, a_1, \dots, a_{n-1} \in \mathbb{N}$, consider the monomial ideal $L(a_0, a_1, \dots, a_{n-1}) \subset S$ from [RS97, Notation 1.2] with generators

$$\langle x_0^{a_{n-1}+1}, x_0^{a_{n-1}} x_1^{a_{n-2}+1}, \dots, x_0^{a_{n-1}} x_1^{a_{n-2}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, x_0^{a_{n-1}} x_1^{a_{n-2}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle.$$

These are our saturated lexicographic ideals.

Lemma 4.2. *Let $\mathsf{p}(t) = \sum_{j=1}^{r} \binom{t+b_j-j+1}{b_j}$ have Gotzmann partition $(b_1, b_2, \cdots, b_r)$, let a_j be the number of parts equal to j, and let $n \in \mathbb{N}$ satisfy $n > d := \deg \mathsf{p}$.*

(i) We have

$$\begin{aligned} L_n^{\mathsf{p}} &= L(a_0, a_1, \dots, a_{n-1}) \\ &= \langle x_0, x_1, \dots, x_{n-d-2}, x_{n-d-1}^{a_d+1}, \\ &\qquad x_{n-d-1}^{a_d} x_{n-d}^{a_{d-1}+1}, \dots, x_{n-d-1}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, x_{n-d-1}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle. \end{aligned}$$

(ii) If there is an integer $0 \le \ell \le d-1$ such that $a_j = 0$ for all $j \le \ell$, and $a_{\ell+1} > 0$, then the minimal monomial generators of L_n^{p} are given by $m_1, m_2, \ldots, m_{n-\ell-1}$, where

$$m_i = x_{i-1}, \text{ for all } 1 \le i \le n-d-1,$$
$$m_{n-d+k} = \left(\prod_{j=0}^{k-1} x_{n-d-1+j}^{a_{d-j}}\right) x_{n-d-1+k}^{a_{d-k}+1}, \text{ for all } 0 \le k \le d-\ell-2, \text{ and}$$
$$m_{n-\ell-1} = \prod_{j=0}^{d-\ell-1} x_{n-d-1+j}^{a_{d-j}}.$$

If $a_0 \neq 0$, then the minimal monomial generators are those listed in (i).

Proof. See [MN14, Theorem 2.3] and [Sta17, Lemma 2.10]. □

Importantly, Lemma 4.2 shows that all sequences $(a_0, a_1, \ldots, a_{n-1})$ of nonnegative integers determine a lexicographic ideal.

The two operations A and Φ on admissible Hilbert polynomials have analogues on lexicographic ideals. For any ideal $I \subseteq S$, we use $\Phi(I) := I \cdot S[x_{n+1}]$ to denote the lifted ideal.

Corollary 4.3. *Let p be admissible, $n > \deg \mathsf{p}$ an integer, and $L_n^{\mathsf{p}} = L(a_0, a_1, \ldots, a_{n-1})$. We have $A(L_n^{\mathsf{p}}) := L_n^{A(\mathsf{p})} = L(a_0+1, a_1, a_2, \ldots, a_{n-1})$ and $\Phi\big(L_n^{\mathsf{p}}\big) = L_{n+1}^{\Phi(\mathsf{p})} = L(0, a_0, a_1, \ldots, a_{n-1})$.*

Proof. These both follow directly from Lemma 4.2(i). Cf. Proposition/Definition 3.1. □

5. Borel-fixed Ideals

Here we review some essential properties of Borel-fixed ideals. An ideal $I \subseteq S$ is ***Borel-fixed*** if it is fixed by the action $\gamma \cdot x_j = \sum_{i=0}^{n} \gamma_{ij} x_i$ of upper triangular matrices $\gamma \in \mathrm{GL}_{n+1}(\mathbb{K})$. Pardue [Par94, II] gives the following combinatorial criterion for I to be Borel-fixed when $\mathbb{K}$ is infinite of characteristic $p > 0$: I is monomial and for each monomial $x^u \in I$, if $x_i \succ x_j$, then $x_j^{-k} x_i^k x^u \in I$ holds, for all $k \le_p u_j$. Here, $k \le_p \ell$ means that in the base-p expansions $k = \sum_i k_i p^i, \ell = \sum_i \ell_i p^i$, we have $k_i \le \ell_i$, for all i.

When $\mathbb{K}$ has characteristic 0, Pardue's criterion reduces to the following property (which makes sense in any characteristic). A monomial ideal $I \subseteq S$ is ***strongly stable***, or 0-***Borel***, if, for all monomials $m \in I$, for all $x_j | m$, and for all $x_i \succ x_j$, we have $x_j^{-1} x_i m \in I$. A strongly stable ideal is always Borel-fixed, but the converse is not true in positive characteristic. A Borel-fixed ideal that is not strongly stable is called a ***nonstandard*** Borel-fixed ideal.

For a monomial m, let $\max m$ be the maximum index j such that $x_j | m$, and $\min m$ be the minimum such index.

Example 5.1. The monomial ideal $I = \langle x_0^2, x_0 x_1, x_1^2 \rangle \subset S$ is strongly stable, where $n \ge 1$. The ideal $I := \langle x_0^p, x_1^p \rangle \subset S$ is a nonstandard Borel-fixed ideal when $\mathbb{K}$ has characteristic $p > 0$. The monomial $m = x_1^5 x_2 x_7^2 \in \mathbb{K}[x_0, x_1, \ldots, x_{13}]$ satisfies $\max m = 7$ and $\min m = 1$.

For a monomial ideal I, let $G(I)$ denote its minimal set of monomial generators. Whether I is Borel-fixed can be checked on monomials in $G(I)$; see [Eis95, § 15.9.3].

Saturated strongly stable ideals are generated by expansions and lifting. Let $I \subseteq S$ be a saturated strongly stable ideal. A generator $g \in G(I)$ is ***expandable*** if there are no elements of $G(I)$ in the set $\{x_i^{-1}x_{i+1}g \mid x_i \text{ divides } g \text{ and } 0 \le i < n-1\}$. The ***expansion*** of I at an expandable generator g is the monomial ideal $I' \subseteq S$ with minimal generators

$$G(I') := (G(I) \setminus \{g\}) \cup \{gx_j \mid \max g \le j \le n-1\};$$

see [MN14, Definition 3.4]. The monomial $1 \in \langle 1 \rangle$ is vacuously expandable with expansion $\mathfrak{m} \subset S$. The expansion of a saturated strongly stable ideal is again strongly stable, by definition. It is also saturated, as we see in Lemma 5.3.

Example 5.2. Let $L_n^{\mathsf{p}} = L(a_0, a_1, \ldots, a_{n-1}) = \langle m_1, m_2, \ldots, m_{n-\ell-1}\rangle$, as in Lemma 4.2. The expansion at $m_{n-\ell-1}$ is easily verified to be $L(a_0+1, a_1, a_2, \ldots, a_{n-1}) = A\big(L_n^{\mathsf{p}}\big)$.

For a Borel-fixed ideal $I \subseteq S$, let $\nabla(I) \subseteq \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$ be the image of I under the mapping $S = \mathbb{K}[x_0, x_1, \ldots, x_n] \to \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$, defined by $x_j \mapsto x_j$ for $0 \le j \le n-2$ and $x_j \mapsto 1$ for $n-1 \le j \le n$. The following lemma shows how this relates to saturation and generalizes Corollary 4.3 to Borel-fixed ideals.

Lemma 5.3. *Let $I \subseteq S$ be a Borel-fixed ideal over an infinite field.*

(i) We have $(I : \mathfrak{m}_k^\infty) = (I : x_k^\infty)$, where $\mathfrak{m}_k := \langle x_0, x_1, \ldots, x_k\rangle \subset S$, for all $0 \le k \le n$.
(ii) If I is saturated, then $\mathsf{p}_{\nabla(I)} = \nabla(\mathsf{p}_I) := \mathsf{p}_I(t) - \mathsf{p}_I(t-1)$ holds.
(iii) If I is saturated, then there exists $j \in \mathbb{N}$ such that $\mathsf{p}_{\Phi(I)} = A^j\Phi(\mathsf{p}_I)$.
(iv) For a saturated strongly stable I with expansion I', we have $\mathsf{p}_{I'} = A(\mathsf{p}_I) := 1 + \mathsf{p}_I$.

Proof.

(i) See [Par94, II, Proposition 9].
(ii) Let $J = I \cap \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]$. Because I is saturated, x_n is not a zero-divisor and

$$0 \longrightarrow (S/I)(-1) \longrightarrow S/I \longrightarrow \mathbb{K}[x_0, x_1, \ldots, x_{n-1}]/J \longrightarrow 0$$

is a short exact sequence. The Hilbert function of J now satisfies $\mathsf{h}_J(i) = \mathsf{h}_I(i) - \mathsf{h}_I(i-1)$, for all $i \in \mathbb{Z}$. Saturating J with respect to $\langle x_0, x_1, \ldots, x_{n-1}\rangle$ gives $\nabla(I)$, so we find that $\mathsf{p}_{\nabla(I)}(t) = \mathsf{p}_I(t) - \mathsf{p}_I(t-1) = \nabla(\mathsf{p}_I)(t)$.
(iii) The ideal $\Phi(I)$ is saturated by (i) and Borel-fixed by Pardue's criterion, with no elements of $G(\Phi(I))$ divisible by x_n. Thus, $\nabla\big(\Phi(I)\big) = I$. Then $\nabla(\mathsf{p}_{\Phi(I)}) = \mathsf{p}_I$ follows by (ii), so $\Phi\nabla(\mathsf{p}_{\Phi(I)}) = \Phi(\mathsf{p}_I)$. Now [Sta17, Lemma 2.5] shows $\mathsf{p}_{\Phi(I)} - \Phi\nabla(\mathsf{p}_{\Phi(I)}) \in \mathbb{N}$. ☐
(iv) See [MN14, Lemma 3.15].

Lemma 5.3 highlights some important properties that are used together with expansions to generate all saturated strongly stable ideals. However, we also wish to understand nonstandard Borel-fixed ideals. Fortunately, other than (iv), these properties hold for all Borel-fixed ideals and are useful in Section 6.

An algorithm for generating saturated strongly stable ideals was first developed by Reeves in [Ree92]; we follow the approach of [MN14]. The heart of Reeves' algorithm is the following theorem.

Theorem 5.4. *If I is a saturated strongly stable ideal of codimension c, then there is a finite sequence $I_{(0)}, I_{(1)}, \ldots, I_{(i)}$ such that $I_{(0)} = \langle 1 \rangle = \mathbb{K}[x_0, x_1, \ldots, x_c]$, $I_{(i)} = I$, and $I_{(j)}$ is an expansion or lifting of $I_{(j-1)}$, for all $1 \le j \le i$.*

Proof. See [MN14, Theorem 4.4]. ☐

When $I \neq \langle 1 \rangle$, the first step is always to expand $\langle 1 \rangle$ to $\langle x_0, x_1, \dots, x_{c-1} \rangle$. We suppress this step, as we assume $\mathsf{p} \neq 0$. The sequences of expansions and liftings are not generally unique, but Corollary 4.3 shows that they are for lexicographic ideals. Theorem 5.4 leads to the following algorithm; see [Ree92, Appendix A] and [MN14, § 4].

Algorithm 5.5.

Input: an admissible Hilbert polynomial $\mathsf{p} \in \mathbb{Q}[t]$ and $n \in \mathbb{N}$ satisfying $n > \deg \mathsf{p}$
Output: all saturated strongly stable ideals with Hilbert polynomial p in $\mathbb{K}[x_0, x_1, \dots, x_n]$

$j = 0$; $d = \deg \mathsf{p}$;
$\mathsf{q}_0 = \nabla^d(\mathsf{p})$; $\mathsf{q}_1 = \nabla^{d-1}(\mathsf{p})$; $\dots$; $\mathsf{q}_{d-1} = \nabla^1(\mathsf{p})$; $\mathsf{q}_d = \mathsf{p}$;
$\mathcal{S} = \{\langle x_0, x_1, \dots, x_{n-d-1} \rangle\}$, where $\langle x_0, x_1, \dots, x_{n-d-1} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{n-d}]$;
WHILE $j \leq d$ DO
    $\mathcal{T} = \varnothing$;
    FOR $J \in \mathcal{S}$, considered as an ideal in $\mathbb{K}[x_0, x_1, \dots, x_{n-d+j}]$ DO
        IF $\mathsf{q}_j - \mathsf{p}_J \geq 0$ THEN
            compute all sequences of $\mathsf{q}_j - \mathsf{p}_J$ expansions that begin with J;
            $\mathcal{T} = \mathcal{T} \cup$ the resulting set of sat. str. st. ideals with Hilbert polynomial q_j;
    $\mathcal{S} = \mathcal{T}$;
    $j = j + 1$;
RETURN $\mathcal{S}$

Proof. See [MN14, Algorithm 4.6]. Here, $\mathcal{S}$ is reset at the jth step to Moore's $\mathcal{S}^{(d-j)}$. □

6. Classifying Hilbert Schemes with Two Borel-fixed Points

We now prove that the classification of Hilbert schemes with exactly two Borel-fixed points given in [Ram19] holds in all characteristics, after one minor adjustment. There are two main tools used to derive the classification in characteristic 0, namely, Reeves' algorithm and Theorem 3.2. Reeves' algorithm does not produce nonstandard Borel-fixed ideals, so we must alter our approach. Fortunately, the recursive properties of Lemma 5.3 together with the classification in Theorem 3.2 are enough to show that all Borel-fixed points are strongly stable on the Hilbert schemes we consider.

We denote $R := \mathbb{K}[x_0, x_1, \dots, x_{n-1}] \supset \mathfrak{n}_k := \langle x_0, x_1, \dots, x_k \rangle$, for $0 \leq k \leq n-1$. As before, we also denote $d := \deg \mathsf{p}$ and $n := d + c$. We start with some basic facts and only study the case $c \geq 2$, by the reduction in [Ram19, § 2].

Lemma 6.1. *The Hilbert scheme* $\mathrm{Hilb}^{3t+1}(\mathbb{P}^n)$ *has* 3 *Borel-fixed points, for all* $n \geq 3$.

Proof. Suppose $I \subset S$ is a saturated Borel-fixed ideal with Hilbert polynomial $\mathsf{p} = 3t + 1$. So I is generated in degrees up to the Gotzmann number $r = 4$. The ideal $\nabla(I)$ is saturated and Borel-fixed with Hilbert polynomial $\mathsf{p}_{\nabla(I)} = 3$, by Lemma 5.3. Thus, $\nabla(I)$ equals either $\mathfrak{n}_{c-3} + \langle x_{c-2}^2, x_{c-2}x_{c-1}, x_{c-1}^2 \rangle$ or $\mathfrak{n}_{c-2} + \langle x_{c-1}^3 \rangle$, by Theorem 6.3(i)(a) (here $n - 1 = c$). Lemma 5.3(i) shows $\nabla(I) = (J : x_c^\infty)$, where $J := I \cap R$. Examining generators, this means $\Phi(\nabla(I)) = (I : x_{n-1}^\infty) \supseteq I$. When $\nabla(I) = \mathfrak{n}_{c-3} + \langle x_{c-2}^2, x_{c-2}x_{c-1}, x_{c-1}^2 \rangle$ holds, the lift $\Phi(\nabla(I))$ has Hilbert polynomial $3t + 1$ and must equal I. When $\nabla(I)$ is lexicographic, we have $I \subset L := \mathfrak{m}_{n-3} + \langle x_{n-2}^3 \rangle \subset S$. In particular, $I_4 \subset L_4$ has codimension 1. The monomials $x_0 x_n^3, x_1 x_n^3, \dots, x_{n-3} x_n^3, x_{n-2}^3 x_n$ cannot all belong to I_4, as I is saturated, so one must be removed. By Pardue's criterion, the possibilities are to remove $x_{n-3} x_n^3$ or $x_{n-2}^3 x_n$.

If we remove the first, then I contains, and thus equals, the expansion of L at x_{n-3}. If we remove the second, then I equals L_n^{p}. $\square$

The proof of Lemma 6.1 demonstrates some key aspects of the strategy for proving Theorem 1.1. That is, there are limited possibilites for what $\nabla(I)$ can be, which restricts what $(I : x_{n-1}^{\infty})$ can be. Then, because we are seeking Hilbert schemes with few Borel-fixed points, the Hilbert polynomials of I and $(I : x_{n-1}^{\infty})$ can only differ by a small constant. This, in turn, means that I can be effectively derived from the ideal $(I : x_{n-1}^{\infty})$ using Pardue's criterion.

We need one last lemma before proving the main theorem.

Lemma 6.2. *Let $n = d + c$, for $c \geq 2$. If $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ has exactly two Borel-fixed points, then $L_n^{\mathsf{p}-1}$ is either $\mathfrak{m}_{c-2} + \langle x_{c-1}^a \rangle$, for some $a > 1$, or $\mathfrak{m}_{c-2} + x_{c-1}^a \langle x_{c-1}, x_c, \ldots, x_{n'-1} \rangle$, for some $a > 0$ and $c < n' \leq n$.*

Proof. By Theorem 3.2, $\mathsf{p}-1$ is admissible. Then Algorithm 5.5 implies that $L_n^{\mathsf{p}-1}$ is generated in at most two degrees; see [Sta17, Proposition 4.4]. If $L_n^{\mathsf{p}-1}$ is generated in a single degree, then it must equal $\mathfrak{m}_{c-1} \subset S$, by Lemma 4.2. Then Theorem 3.2 implies $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ has a unique Borel-fixed point. Lemma 4.2 now leaves the provided options. $\square$

Using this preliminary classification, we can now prove the main theorem.

Theorem 6.3. *Let $n = d + c$, where $d = \deg \mathsf{p}$ and $c \geq 2$. The Hilbert scheme $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ has exactly two Borel-fixed ideals if and only if one of the following conditions holds:*

- *(i)* *(a)* $b_1 = 0$ *and* $r = 3$,
  - *(a')* $b_1 = 0$ *and* $r = 4$, *if* $n = 2$ *and* $\mathbb{K}$ *does not have characteristic* 2,
  - *(b)* $b_1 = \cdots = b_{r-1} = 1$ *and* $b_r = 0$, *for* $r - 1 \neq 1, 3$,
  - *(c)* $b_1 = \cdots = b_{r-1} \geq 2$ *and* $b_r = 0$, *for* $r - 1 \neq 1$,
- *(ii)* *(a)* $b_1 > b_2 = 0$ *and* $r = 3$,
  - *(b)* $b_1 = \cdots = b_{r-2} > b_{r-1} = 1$ *and* $b_r = 0$, *for* $r - 2 \neq 2$, *and*
  - *(c)* $b_1 = \cdots = b_{r-2} > b_{r-1} \geq 2$ *and* $b_r = 0$.

Proof. Let $I \subset S$ be saturated and Borel-fixed, whose Hilbert polynomial p has Gotzmann partition $b := (b_1, b_2, \ldots, b_r)$. If $[I]$ lies on a $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ having two Borel-fixed ideals, then either $a = r - 1$ and $b_1 = \cdots = b_a \geq b_r = 0$, or $a = r - 2$ and $b_1 = \cdots = b_a > b_{a+1} \geq b_r = 0$, by Lemma 6.2. When b does not satisfy any cases above, either I is lexicographic, by Theorem 3.2, or there are at least three saturated strongly stable ideals with Hilbert polynomial p, by Reeves' algorithm, and these are Borel-fixed. Thus, we suppose I satisfies one of the cases. The analysis of several cases is analogous to the proof of Lemma 6.1.

(i)(a). Here $b = (0, 0, 0)$, i.e. $\mathsf{p} = 3$ and $n = c$. We know that I is generated in degrees up to $r = 3$. Suppose $n = 2$, so $I \subset S = \mathbb{K}[x_0, x_1, x_2]$. Then $x_1^N \in I$, for some minimal $N \leq 3$, by [Par94, II, Corollary 8]. If $N = 1$, then $x_0 \in I$, which implies $\mathsf{p} = 1$. If $N = 3$, then $x_1^2 x_2^{j-2}, x_1 x_2^{j-1}, x_2^j \notin I_j$, for all $j \gg 0$, so $x_0 \in I$ and $I = \langle x_0, x_1^3 \rangle$ is lexicographic. If $N = 2$, then $x_0^2 \in I$. Thus one of the monomials $x_0 x_1 x_2, x_0 x_2^2, x_1 x_2^2, x_2^3$ must belong to I_3. As I is saturated, we can only have $x_0 x_1 x_2 \in I$ and so $I = \langle x_0^2, x_0 x_1, x_1^2 \rangle$ is strongly stable. Now suppose $n > 2$ and $x_0 \notin I$. Then Pardue's criterion implies $x_0 x_n^{j-1}, x_1 x_n^{j-1}, \ldots, x_n^j \notin I_j$, for all $j \gg 0$, giving a contradiction. So $x_0 \in I$ and we are finished, by induction on n.

(i)(a'). Here $b = (0, 0, 0, 0)$, so $\mathsf{p} = 4$ and $n = 2$. As in (i)(a), we can see that I contains one of x_1^4, x_1^3, or x_1^2. Let us summarize: If $x_1^4 \in I$, then $I = \langle x_0, x_1^4 \rangle$ must be lexicographic; if $x_1^3 \in I$, then we must have $I = \langle x_0^2, x_0x_1, x_1^3\rangle$; and if $x_1^2 \in I$, then we can only have $I = \langle x_0^2, x_1^2\rangle$, which is Borel-fixed if and only if $\mathbb{K}$ has characteristic 2.

(i)(b). Here $b = (1, 1, \ldots, 1, 0)$ with $a = r - 1 \neq 1, 3$, so that $\mathsf{p}(t) = at + 2 - \frac{(a-1)(a-2)}{2}$ and $\nabla(\mathsf{p}) = a$. We study I by lifting $\nabla(I)$. If $a = 2$, then the result follows as in Lemma 6.1, so let $a \geq 4$. The goal is to show that if $J \subset R$ is a nonlexicographic saturated Borel-fixed ideal with Hilbert polynomial a, then $\mathsf{p}_{\Phi(J)} - \mathsf{p} > 0$. This holds for strongly stable J, by [Ram19, Theorem 2.16], so we assume that J is nonstandard. As J is nonlexicographic, we have $x_{c-1}^N \in G(J)$, for some $2 \leq N < a$, and also $x_{c-2} \notin J$. Consider the minimal strongly stable ideal $J^0 \supset J$; we obtain J^0 from J by keeping the generators of J and adding all "missing" monomials $x_j^{-1}x_ih \notin J$, where $h \in G(J)$, $i < j$, and $x_j | h$. Let $a' := \mathsf{p}_{J^0}$, so that $a' < a$. Because $x_{c-2} \notin J$, it follows that J^0 is not lexicographic, which further shows $N < a'$.

Fix $j \gg 0$. We have $x_{c-1}^{N-1}x_c^{j-N+1}, x_{c-1}^{N-2}x_c^{j-N+2}, \ldots, x_c^j \notin J_j^0$. As J^0 is strongly stable and contains x_{c-1}^N, R_j/J_j^0 is spanned by $(x_{c-1}^{N-i}x_c^{j-N+i}, m_kx_c^{j-\deg m_k} \mid 1 \leq i \leq N, 1 \leq k \leq a' - N)$, where the monomials $m_1, m_2, \ldots, m_{a'-N}$ are not divisible by x_c and $\deg m_k < N$, for all $1 \leq k \leq a' - N$. To describe R_j/J_j, we need a further $a - a'$ monomials from $J_j^0 \setminus J_j$. First, as J^0 is strongly stable, it is generated in degrees up to N. The regularity of J is at most a, so there are $a - a'$ monomials in $J_a^0 \setminus J_a$. We need to alter these monomials to show their effect on the Hilbert function $\mathsf{h}_{\Phi(J)}$. Each has a unique factorization $h = gh'$, where $g \in G(J^0) \setminus J$ and $\max g \leq \min h'$; see [MS05, Lemma 2.11]. Let $h_0, h_1, \ldots, h_{i_g}$ list those whose factorization involves a fixed $g \in G(J^0) \setminus J$ and write $h_i := gp_ix_c^{a-\deg g-\deg p_i}$, where x_c does not divide p_i. Order these such that $h_0 := gx_c^{a-\deg g}$ and $0 < \deg p_1 \leq \deg p_2 \leq \cdots \leq \deg p_{i_g}$. Let p_1' be any monomial dividing p_1 with $\deg p_1' = 1$. Then $f_1 := gp_1' \in J^0 \setminus J$ holds. For $1 < i \leq i_g$, if $\deg p_i < i$, then set $p_i' := p_i$, otherwise let p_i' divide p_i with $\deg p_i' = i$. Then we have $f_i := gp_i' \in J^0 \setminus J$ and f_i differs from the previous monomials $f_0 := g, f_1, \ldots, f_{i-1}$. We write f_i^g to distinguish the monomials obtained from a fixed g. By uniqueness of the factorizations, we have $f_i^g \neq f_{i'}^{g'}$ when $g \neq g'$.

Lifting to S, we see that $S_j \setminus J_j$ contains the monomials $x_{n-2}^k\langle x_{n-1}, x_n\rangle^{j-k}$, for $0 \leq k \leq N-1$, along with $m_k\langle x_{n-1}, x_n\rangle^{j-\deg m_k}$, for $1 \leq k \leq a'-N$, and $f_k^g\langle x_{n-1}, x_n\rangle^{j-\deg f_k^g}$, for $g \in G(J^0) \setminus J$ and $1 \leq k \leq i_g$. This implies

$$\begin{aligned}\mathsf{h}_{\Phi(J)}(j) &\geq \sum_{k=0}^{N-1}(j-k+1) + \sum_{k=1}^{a'-N}(j - \deg m_k + 1) + \sum_{g \in G(J^0)\setminus J}\sum_{k=1}^{i_g}(j - \deg f_k^g + 1) \\ &\geq \sum_{i=0}^{N-1}(j-i+1) + \sum_{i=N}^{a'-1}(j-N+1+1) + \sum_{i=a'}^{a-1}(j-i+1+1) \\ &\geq aj + 1 - \frac{(a-1)(a-2)}{2} + 2 \geq \mathsf{p}(j) + 1,\end{aligned}$$

which shows that $\mathsf{p}_{\Phi(J)} - \mathsf{p} > 0$, as desired.

(i)(c). Here $b = (d, d, \ldots, d, 0)$, $d \geq 2$, and $a = r - 1 \geq 2$. Lemma 5.3 and Theorem 3.2 imply $\nabla(I) = L_{n-1}^{\nabla(\mathsf{p})} = \mathfrak{n}_{c-2} + \langle x_{c-1}^a\rangle \subset R$. It follows that $L := L_n^{\mathsf{p}-1} = (I : x_{n-1}^\infty)$ and $I_{a+1} \subset L_{a+1}$ has codimension 1. The monomials $x_0x_n^a, x_1x_n^a, \ldots, x_{c-2}x_n^a, x_{c-1}^ax_n \in L_{a+1}$ cannot all belong to I. Pardue's criterion implies either $x_{c-2}x_n^a \notin I_{a+1}$ or $x_{c-1}^ax_n \notin I_{a+1}$. In the first case, I is the expansion of L at x_{c-2} and in the second case, I is the expansion of L at x_{c-1}^a.

(ii)(a). Here $b = (d, 0, 0)$ with $d > 0$, so $\mathsf{p} = \binom{t+d}{d} + 2$. Lemma 5.3 and Theorem 3.2 imply that $\nabla(I) = L_{n-1}^{\nabla(\mathsf{p})} = \mathfrak{n}_{c-1}$. This implies $L := L_n^{\mathsf{p}-2} = \mathfrak{m}_{c-1} = (I : x_{n-1}^{\infty}) \supset I$, so I_3 has codimension 2 in L_3. As I is saturated, if $x_{c-1}x_n^2 \in I$, then $I \supseteq L$ holds. So $x_{c-1}x_n^2 \notin I$. Now suppose I contains $x_{c-2}x_n^2$. Then $\mathfrak{m}_{c-2} \subset I$ holds. If we also have $x_{c-1}x_{n-1}x_n \in I$, then $I \supseteq L_n^{\mathsf{p}-1}$ must hold, which is nonsense. Thus, if $x_{c-2}x_n^2 \in I$, then I_3 contains all monomials in L_3 except $x_{c-1}x_{n-1}x_n, x_{c-1}x_n^2$. This implies that $I = L_n^{\mathsf{p}}$. Now suppose I does not contain $x_{c-2}x_n^2$. Then I equals the expansion of $L_n^{\mathsf{p}-1}$ at x_{c-2}. If $a \neq 1$ and $b_{a+1} = 0$, then Reeves' algorithm generates a third saturated Borel-fixed ideal; see [Ram19, Theorem 2.16].

(ii)(b). Here $b = (d, d, \ldots, d, 1, 0)$ with $d > 1$ and $a = r - 2 \neq 2$. If $a = 1$, then $\nabla(I) = L_{n-1}^{\nabla(I)}$, by Lemma 5.3 and Theorem 3.2. Setting $L := \mathfrak{m}_{c-2} + x_{c-1}\langle x_{c-1}, x_c, \ldots, x_{n-2}\rangle$, we see that $I_3 \subset L_3$ is a codimension 1 subspace. The elements

$$x_0x_n^2, x_1x_n^2, \ldots, x_{c-2}x_n^2, x_{c-2}^2x_n, x_{c-1}x_cx_n, \ldots, x_{c-1}x_{n-2}x_n$$

cannot all belong to I_3. By Pardue's criterion, either $x_{c-2}x_n^2$ or $x_{c-1}x_{n-2}x_n$ must be removed. Removing $x_{c-2}x_n^2$ implies I is the expansion of L at x_{c-2}. Removing $x_{c-1}x_{n-2}x_n$ implies I is the expansion of L at $x_{c-1}x_{n-2}$.

Suppose $a = 3$. If $d = 2$, then there are three saturated Borel-fixed ideals with Hilbert polynomial $\nabla(\mathsf{p}) = 3t+1$ and codimension c, by Lemma 6.1. These are all strongly stable and one can directly verify that the lifts of the nonlexicographic ones have Hilbert polynomial $\mathsf{p}+1$. Thus, $\nabla(I) = L_{n-1}^{3t+1}$ holds and $I_5 \subset L_5$ has codimension 1, where $L = \mathfrak{m}_{c-2} + \langle x_{c-1}^4, x_{c-1}^3x_c\rangle$. Again, Pardue's criterion tells us that I is an expansion of L. If $d \geq 3$, then $\nabla(I)$ falls into case (i)(c). Lifting the nonlexicographic Borel-fixed ideal gives a Hilbert polynomial that is too big, by [Ram19]. Repeating the previous argument then shows that I is strongly stable. If $a \geq 4$, then $\nabla(I)$ falls into cases (i)(b)-(c), so again must be lexicographic and I must be strongly stable.

(ii)(c). Here $b = (d, d, \ldots, d, b_{a+1}, 0)$ with $d > b_{a+1} \geq 2$. Lemma 5.3 and Theorem 3.2 imply that $\nabla(I) = L_{n-1}^{\nabla(\mathsf{p})}$ and setting $L := L_n^{\mathsf{p}-1} = \mathfrak{m}_{c-2} + x_{c-1}^a\langle x_{c-1}, x_c, \ldots, x_{n'-1}\rangle$, where $c < n' < n$ is as in Lemma 6.2, we see that $I_{a+2} \subset L_{a+2}$ has codimension 1. The elements

$$x_0x_n^{a+1}, x_1x_n^{a+1}, \ldots, x_{c-2}x_n^{a+1}, x_{c-1}^{a+1}x_n, x_{c-1}^a x_c x_n, \ldots, x_{c-1}^a x_{n'-1}x_n$$

cannot all belong to I, so Pardue's criterion implies $x_{c-2}x_n^{a+1} \notin I$ or $x_{c-1}^a x_{n'-1}x_n \notin I$. The first choice implies I is the expansion of L at x_{c-2}. The second implies that I is lexicographic. $\square$

Proof of Theorem 1.1. Theorem 6.3 proves the claim. $\square$

Remark 6.4. Continuing with this method of adapting Reeves' algorithm to the Borel-fixed case, one can understand the Borel-fixed points on various Hilbert schemes with more than two Borel-fixed points in arbitrary characteristic. For example, one can prove that the Hilbert scheme $\mathrm{Hilb}^{\mathsf{p}}(\mathbb{P}^n)$ has exactly 3 Borel-fixed points, where p has Gotzmann partition $(d, d, 1, 0)$ with $d > 1$; cf. [Ram19, § 4].

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