QUEEN'S UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 212
MIDTERM EXAMINATION
VERSION B
10 FEBRUARY 2016
PROFESSOR: ANDREW STAAL

Name:

Student Number:

- This examination is *two hours* in length.
- Calculators, data sheets, or other aids are *not* permitted.
- Each question is worth 10 points.
- To receive full credit, you must explain your answers.
- Answers are to be recorded on the question paper
  (use the backs of pages if necessary).

1	2	3	4	5	total

1. Consider the linear transformation $Q\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $Q(a_1, a_2) = (2a_1 + a_2, 2a_2)$.

Alternate 1. Consider the linear transformation $R\colon \mathbb{Q}^2 \to \mathbb{Q}^2$ defined by $R(b_1, b_2) = (3b_1, b_1 + 3a_2)$.

(a) Prove that the subspace $U = \operatorname{span}\big((1, 0)\big) \subset \mathbb{R}^2$ is invariant under Q.

Solution. We have to check that for every $u \in U$, we have $Q(u) \in U$. Any vector $u \in U$ has the form $(c, 0)$ for $c \in \mathbb{F}$, and we have $Q(u) = Q(c, 0) = (2c + 0, 2 \cdot 0) = (2c, 0) \in U$. Thus, U is Q-invariant. □

Alternate (a) Prove that the subspace $U = \operatorname{span}\big((0, 1)\big) \subset \mathbb{Q}^2$ is invariant under R.

Solution. We have to check that for every $u \in U$, we have $R(u) \in U$. Any vector $u \in U$ has the form $(0, c)$ for $c \in \mathbb{F}$, and we have $R(u) = R(0, c) = (3 \cdot 0, 0 + 3c) = (0, 3c) \in U$. Thus, U is R-invariant. □

(b) Show that there does not exist a Q-invariant subspace W such that $\mathbb{R}^2 = U \oplus W$.

Solution. We prove this by contradiction. If there exists such a W, then we have $2 = \dim(\mathbb{R}^2) = \dim(U) + \dim(W) = 1 + \dim(W)$, so $\dim(W) = 1$ and W is spanned by any nonzero $w \in W$. Such a w is an eigenvector, because $Q(w) \in W = \operatorname{span}(w)$. Hence, if $w = (w_1, w_2)$, then for some $\lambda \in \mathbb{F}$, we have $\lambda w = (\lambda w_1, \lambda w_2) = Q(w) = (2w_1 + w_2, 2w_2)$. As $w \notin U$, we know $w_2 \neq 0$, which means that $\lambda = 2$. Thus, $2w_1 = 2w_1 + w_2$, which implies $w_2 = 0$, a contradiction. □

Alternate (b) Show that there does not exist a R-invariant subspace W such that $\mathbb{Q}^2 = U \oplus W$.

Solution. We prove this by contradiction. If there exists such a W, then we have $2 = \dim(\mathbb{R}^2) = \dim(U) + \dim(W) = 1 + \dim(W)$, so $\dim(W) = 1$ and W is spanned by any nonzero $w \in W$. Such a w is an eigenvector, because $R(w) \in W = \operatorname{span}(w)$. Hence, if $w = (w_1, w_2)$, then for some $\lambda \in \mathbb{F}$, we have $\lambda w = (\lambda w_1, \lambda w_2) = R(w) = (3w_1, w_1 + 3w_2)$. As $w \notin U$, we know $w_1 \neq 0$, which means that $\lambda = 3$. Thus, $3w_2 = w_1 + 3w_2$, which implies $w_1 = 0$, a contradiction. □

2. Consider the linear transformation $T\colon \mathbb{F}^2 \to \mathbb{F}^2$ defined by $T(x_1, x_2) = (x_2, -x_1)$.

Alternate 2. Consider the linear transformation $T\colon \mathbb{F}^2 \to \mathbb{F}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$.

(a) Suppose that $\mathbb{F} = \mathbb{R}$. List all T-invariant subspaces of $\mathbb{R}^2$.

Solution. This map is clockwise rotation by $90°$, given by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. As we have seen in class, such a rotation of $\mathbb{R}^2$ has only trivial invariant subspaces. Thus, the T-invariant subspaces of $\mathbb{R}^2$ are 0 and $\mathbb{R}^2$ itself. □

Alternate (a) Suppose that $\mathbb{F} = \mathbb{R}$. List all T-invariant subspaces of $\mathbb{R}^2$.

Solution. This map is counterclockwise rotation by $90°$, given by the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. As we have seen in class, such a rotation of $\mathbb{R}^2$ has only trivial invariant subspaces. Thus, the T-invariant subspaces of $\mathbb{R}^2$ are 0 and $\mathbb{R}^2$ itself. □

(b) Suppose that $\mathbb{F} = \mathbb{C}$. Compute all eigenvalues of T. For each eigenvalue, give a corresponding eigenvector.

Solution. Suppose that $v = (v_1, v_2)$ is an eigenvector, with eigenvalue λ. We have $(\lambda v_1, \lambda v_2) = (v_2, -v_1)$. Substituting, $-v_1 = \lambda v_2 = \lambda(\lambda v_1)$, which gives $\lambda^2 + 1 = 0$ (or $v_1 = 0$, which implies $v = 0$, a contradiction). Thus, $\lambda = \pm i$.
At this point, for $\lambda = i$, we can solve the system $i(a+bi) = c+di$, $i(c+di) = -a-bi$, where $v_1 = a+bi$ and $v_2 = c+di$, and similarly for $\lambda = -i$. But we have seen a similar question in class, so it makes sense to make similar guesses for eigenvectors, like $(1, i)$ and $(1, -i)$. Checking these, we have $i(1, i) = (i, -1) = T(1, i)$, and $-i(1, -i) = (-i, -1) = T(1, -i)$, so these are respective eigenvectors. □

Alternate (b) Suppose that $\mathbb{F} = \mathbb{C}$. Compute all eigenvalues of T. For each eigenvalue, give a corresponding eigenvector.

Solution. Suppose that $v = (v_1, v_2)$ is an eigenvector, with eigenvalue λ. We have $(\lambda v_1, \lambda v_2) = (-v_2, v_1)$. Substituting, $-v_2 = \lambda v_1 = \lambda(\lambda v_2)$, which gives $\lambda^2 + 1 = 0$ (or $v_2 = 0$, which implies $v = 0$, a contradiction). Thus, $\lambda = \pm i$.
At this point, for $\lambda = i$, we can solve the system $i(a+bi) = -c-di$, $i(c+di) = a+bi$, where $v_1 = a+bi$ and $v_2 = c+di$, and similarly for $\lambda = -i$. But we have seen this question in class, and have found the eigenvectors $(i, 1)$ and $(-i, 1)$. Checking these, we have $i(i, 1) = (-1, i) = T(i, 1)$, and $-i(-i, 1) = (-1, -i) = T(-i, 1)$, so these are respective eigenvectors. □

(c) Find a basis for which the matrix $\mathcal{M}(T)$ associated to T is upper-triangular. If possible, give a basis for which $\mathcal{M}(T)$ is diagonal; otherwise explain why this is not possible.

Solution. Neither is possible over $\mathbb{R}$, as this would imply the existence of an eigenvector. With respect to the basis $\big((1, i), (1, -i)\big)$ of $\mathbb{C}^2$, we have $\mathcal{M}(T) = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$, which is both upper triangular and diagonal. □

Alternate (c) Find a basis for which the matrix $\mathcal{M}(T)$ associated to T is upper-triangular. If possible, give a basis for which $\mathcal{M}(T)$ is diagonal; otherwise explain why this is not possible.

Solution. Neither is possible over $\mathbb{R}$, as this would imply the existence of an eigenvector. With respect to the basis $\big((i, 1), (-i, 1)\big)$ of $\mathbb{C}^2$, we have $\mathcal{M}(T) = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$, which is both upper triangular and diagonal. □

3. **(a)** Suppose V is a finite dimensional vector space, $L \in \mathrm{End}(V)$ has $\dim V$ distinct eigenvalues, and $K \in \mathrm{End}(V)$ has the same eigenvectors as L (but not necessarily the same eigenvalues). Prove that $KL = LK$.

Solution. We have seen that eigenvectors belonging to distinct eigenvalues are linearly independent, which implies that the eigenvectors of L form an eigenbasis, say $(v_1, v_2, \ldots, v_{\dim V})$, with corresponding eigenvalues $\lambda_j \in \mathbb{F}$. For each j, we also have $K(v_j) = \kappa_j v_j$ for some $\kappa_j \in \mathbb{F}$. Thus, $KL(v_j) = K(L(v_j)) = K(\lambda_j v_j) = \lambda_j K(v_j) = \lambda_j \kappa_j v_j$. Similarly, $LK(v_j) = \kappa_j \lambda_j v_j$. Hence, $KL(v_j) = LK(v_j)$ for every v_j in the basis, and so $KL = LK$. $\square$

(b) Suppose that $K, L \in \mathrm{End}(V)$ are such that $KL = LK$. Prove that $\mathrm{Ker}(L - \lambda I)$ is invariant under K for every $\lambda \in \mathbb{F}$.

Solution. Let $v \in \mathrm{Ker}(L - \lambda I)$. We want to show that $K(v) \in \mathrm{Ker}(L - \lambda I)$, in other words, that $(L - \lambda I)(K(v)) = 0$. We evaluate $(L - \lambda I)(K(v)) = LK(v) - \lambda K(v) = KL(v) - \lambda K(v) = K(Lv - \lambda v) = K(L - \lambda I)(v) = K(0) = 0$. Hence, $\mathrm{Ker}(L - \lambda I)$ is K-invariant. $\square$

4. The *Fibonacci sequence* $f_1, f_2, \ldots$ is the sequence of integers defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Define a linear operator $F \in \mathrm{End}(\mathbb{R}^2)$ by $F(x, y) = (y, y + x)$.

Alternate 4. The *Lucas sequence* $\ell_1, \ell_2, \ldots$ is the sequence of integers defined by $\ell_1 = 2$, $\ell_2 = 1$, and $\ell_n = \ell_{n-1} + \ell_{n-2}$ for $n \geq 3$. Define a linear operator $L \in \mathrm{End}(\mathbb{R}^2)$ by $L(x, y) = (y, y + x)$.

(a) Compute f_5. Evaluate $F^4(0, 1)$. Explain the relationship between f_5 and $F^4(0, 1)$.

Solution. Computing terms in the sequence gives $f_3 = f_2 + f_1 = 1 + 1 = 2$, $f_4 = f_3 + f_2 = 2 + 1 = 3$, and $f_5 = f_4 + f_3 = 3 + 2 = 5$. On the other hand, $F(0, 1) = (1, 1)$, $F^2(0, 1) = F(1, 1) = (1, 2)$, $F^3(0, 1) = F(1, 2) = (2, 3)$, and $F^4(0, 1) = F(2, 3) = (3, 5)$. Hence, f_5 is the second coordinate of $F^4(0, 1)$. □

Alternate (a) Compute ℓ_5. Evaluate $L^4(-1, 2)$. Explain the relationship between ℓ_5 and $L^4(-1, 2)$.

Solution. Computing terms in the sequence gives $\ell_3 = \ell_2 + \ell_1 = 1 + 2 = 3$, $\ell_4 = \ell_3 + \ell_2 = 3 + 1 = 4$, and $\ell_5 = \ell_4 + \ell_3 = 4 + 3 = 7$. On the other hand, $L(-1, 2) = (2, 1)$, $L^2(-1, 2) = L(2, 1) = (1, 3)$, $L^3(-1, 2) = L(1, 3) = (3, 4)$, and $L^4(-1, 2) = L(3, 4) = (4, 7)$. Hence, ℓ_5 is the second coordinate of $L^4(0, 1)$. □

(b) Prove by induction that $F^n(0, 1) = (f_n, f_{n+1})$ for every $n \in \mathbb{N}^*$.

Solution. If $n = 1$, then $F^1(0, 1) = F(0, 1) = (1, 1) = (f_1, f_2)$, which proves the base case. For the inductive step, suppose that $F^{n-1}(0, 1) = (f_{n-1}, f_n)$. Thus, $F^n(0, 1) = F(F^{n-1}(0, 1)) = F(f_{n-1}, f_n) = (f_n, f_n + f_{n-1}) = (f_n, f_{n+1})$ by definition of the Fibonacci sequence, as desired. □

Alternate (b) Prove by induction that $L^n(-1, 2) = (\ell_n, \ell_{n+1})$ for every $n \in \mathbb{N}^*$.

Solution. If $n = 1$, then $L^1(-1, 2) = L(-1, 2) = (2, 1) = (\ell_1, \ell_2)$, which proves the base case. For the inductive step, suppose that $L^{n-1}(-1, 2) = (\ell_{n-1}, \ell_n)$. Thus, $L^n(-1, 2) = L(L^{n-1}(-1, 2)) = L(\ell_{n-1}, \ell_n) = (\ell_n, \ell_n + \ell_{n-1}) = (\ell_n, \ell_{n+1})$ by definition of the Lucas sequence, as desired. □

(c) Determine the eigenvalues of F.

Solution. Let $F(x, y) = \lambda(x, y)$, or equivalently $(y, y + x) = (\lambda x, \lambda y)$. Substituting $y = \lambda x$ into $y + x = \lambda y$, we have $\lambda x + x = \lambda^2 x$, which gives $\lambda^2 - \lambda - 1 = 0$ or $x = 0$. If $x = 0$, then $y = 0$, a contradiction. Thus, $\lambda = \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$. The value $\dfrac{1 + \sqrt{5}}{2}$ is known as the Golden Ratio. □

5. Let V and W be $\mathbb{F}$-vector spaces and let $S \in \mathrm{Hom}(V, W)$.

(a) Suppose that S is injective and $(v_1, v_2, \ldots, v_n)$ is linearly independent in V. Show that $(Sv_1, Sv_2, \ldots, Sv_n)$ is linearly independent in W.

Solution. Suppose that $a_1Sv_1 + a_2Sv_2 + \cdots + a_nSv_n = 0$. We then have $0 = a_1Sv_1 + a_2Sv_2 + \cdots + a_nSv_n = S(a_1v_1 + a_2v_2 + \cdots + a_nv_n)$, which implies $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, as S is injective. Because $(v_1, v_2, \ldots, v_n)$ is linearly independent, $a_1 = a_2 = \cdots = a_n = 0$, so that $(Sv_1, Sv_2, \ldots, Sv_n)$ is linearly independent. □

Alternate (a) Suppose that S is surjective and $(v_1, v_2, \ldots, v_n)$ spans V. Show that $(Sv_1, Sv_2, \ldots, Sv_n)$ spans W.

Solution. Let $w \in W$. Because S is surjective, there exists $v \in V$ such that $Sv = w$. We can write $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, so that $w = Sv = S(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1Sv_1 + a_2Sv_2 + \cdots + a_nSv_n$, which shows that w is in the span of $(Sv_1, Sv_2, \ldots, Sv_n)$, as desired. □

(b) Suppose $A, B \in \mathrm{End}(V)$ and B is invertible. If $g \in \mathbb{F}[t]$ is a polynomial, then show that $g(BAB^{-1}) = Bg(A)B^{-1}$.

Solution. We first use induction to prove this for powers $g = t^n$. If $n = 0$, then $g = 1$, so $g(BAB^{-1}) = I = BIB^{-1}$ (note that $A^0 = I$, by definition). Now suppose that the statement holds for $n = k - 1$, and let $g = t^k$. Thus, $g(BAB^{-1}) = (BAB^{-1})^k = (BAB^{-1})^{k-1}(BAB^{-1}) = (BA^{k-1}B^{-1})(BAB^{-1}) = BA^kB^{-1}$. This proves the statement for powers of t.

Continuing with the general case, let $g(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$. We have $g(BAB^{-1}) = a_0I + a_1(BAB^{-1}) + a_2(BAB^{-1})^2 + \cdots + a_n(BAB^{-1})^n$, which equals $a_0BIB^{-1} + a_1(BAB^{-1}) + a_2(BA^2B^{-1}) + \cdots + a_n(BA^nB^{-1}) = B(a_0I + a_1A + a_2A^2 + \cdots + a_nA^n)B^{-1} = Bg(A)B^{-1}$, as desired. □