Solutions #1

1. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

Solution. Let $X := \{(x, y) \in \mathbb{R}^2 : xy = 0\}$; in other words, X is the union of the x-axis and the y-axis. Since each axis is a line through the origin, it is a subspace. In particular, each axis is closed under scalar multiplication. It follows that X is also closed under scalar multiplication. However, X is not closed under addition; $(1, 0), (0, 1) \in X$ and $(1, 0) + (0, 1) = (1, 1) \notin X$. Therefore, X is not a subspace. □

2. Let $\mathbb{F}$ be any field and let $\mathbb{F}^{\mathbb{F}}$ denote the set of all functions from $\mathbb{F}$ to $\mathbb{F}$. The set $\mathbb{F}^{\mathbb{F}}$ is a vector space over $\mathbb{F}$ with pointwise operations:

$$(f + g)(b) := f(b) + g(b) \qquad\qquad (af)(b) := a\big(f(b)\big)$$

for $f, g \in \mathbb{F}^S$ and $a, b \in \mathbb{F}$. A function $f \in \mathbb{F}^{\mathbb{F}}$ is *even* if $f(-b) = f(b)$ for all $b \in \mathbb{F}$ and *odd* if $f(-b) = -f(b)$ for all $b \in \mathbb{F}$. Prove that the set U_e of all even functions and the set U_o of all odd functions are subspaces of $\mathbb{F}^{\mathbb{F}}$. **Bonus:** Show that $\mathbb{F}^{\mathbb{F}} = U_e \oplus U_o$.

Solution. Let U_e and U_o denote the set of even and odd functions in $\mathbb{F}^{\mathbb{F}}$ respectively. Since the zero function in $\mathbb{F}^{\mathbb{F}}$ is both even and odd, both U_e and U_o are not empty. If $f, g \in U_e$ and $a, b \in \mathbb{F}$, then we have

$$(af + g)(-b) = a\big(f(-b)\big) + g(-b) = a\big(f(b)\big) + g(b) = (af + g)(b),$$

so $af + g \in U_e$. Similarly, if $f, g \in U_o$ and $a, b \in \mathbb{F}$, then we have

$$(af + g)(-b) = a\big(f(-b)\big) + g(-b) = a\big(-f(b)\big) - g(b) = -\big[a\big(f(b)\big) + g(b)\big] = -(af + g)(b)$$

and $af + g \in U_o$. Therefore, both U_e and U_o are subspaces of $\mathbb{F}^{\mathbb{F}}$.

To see that $\mathbb{F}^{\mathbb{F}} = U_e \oplus U_o$, first suppose that $f \in U_e \cap U_o$. For every $b \in \mathbb{F}$, we have $-f(b) = f(-b) = f(b)$, so that $0 = f(b)$. Thus, f is the zero function, and $U_e \cap U_o = \{0\}$. Finally, there is a well-known trick to show that $\mathbb{F}^{\mathbb{F}} = U_e + U_o$. Let $f \in \mathbb{F}^{\mathbb{F}}$; we can always rewrite $f(b)$ as $\frac{1}{2}\left(f(b) + f(-b) + f(b) - f(-b)\right)$. Consider the functions $f_e(b) = \frac{1}{2}\left(f(b) + f(-b)\right)$, and $f_o(b) = \frac{1}{2}\left(f(b) - f(-b)\right)$. It is straightforward to check that $f = f_e + f_o$, $f_e \in U_e$, and $f_o \in U_o$, so that $\mathbb{F}^{\mathbb{F}} = U_e + U_o$. Hence, $\mathbb{F}^{\mathbb{F}} = U_e + U_o$. □

3. Let V be a vector space. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution. Let U_1 and U_2 be subspaces of V.

$\Longrightarrow$ If neither U_1 nor U_2 is contained in the other, then there exist vectors $u_1 \in U_1$ and $u_2 \in U_2$ such that $u_1 \notin U_2$ and $u_2 \notin U_1$. Since U_1 is closed under addition, $u_1 \in U_1$ and $u_2 = (u_1 + u_2) - u_1$, we deduce that $u_1 + u_2 \notin U_1$. By symmetry, we also have $u_1 + u_2 \notin U_2$, which implies that $u_1 + u_2 \notin U_1 \cup U_2$. Hence, $U_1 \cup U_2$ is not closed under addition, so $U_1 \cup U_2$ is not a subspace.

$\Longleftarrow$ If $U_1 \subseteq U_2$, then $U_1 \cup U_2 = U_2$ is a subspace of V. Similarly, if $U_2 \subseteq U_1$, then $U_1 \cup U_2 = U_1$ is a subspace of V. $\square$