Solutions #2

1. The *transpose* A^T of an $(m \times n)$-matrix A is obtained from A by interchanging the rows with the columns; in other words if $A = [a_{i,j}]$ then $A^\mathsf{T} = [a_{j,i}]$. A matrix A is *symmetric* if $A^\mathsf{T} = A$ and *skew-symmetric* if $A^\mathsf{T} = -A$.

(a) Prove that the set W_{skew} of all skew-symmetric $(n \times n)$-matrices is a subspace of $\mathbb{Q}^{n\times n}$.

(b) Let W_{sym} be the subspace of $\mathbb{Q}^{n\times n}$ consisting of all symmetric matrices. Prove that $\mathbb{Q}^{n\times n} = W_{\text{sym}} \oplus W_{\text{skew}}$.

Solution.

(a) Since the zero matrix is equal to the negative of its transpose, we have $0 \in W_{\text{skew}}$ which means that W_{skew} is nonempty. If $A, B \in W_{\text{skew}}$ and $c \in \mathbb{Q}$, then we have

$$(cA + B)^\mathsf{T} = cA^\mathsf{T} + B^\mathsf{T} = c(-A) + (-B) = -(cA + B)$$

and $cA + B \in W_{\text{skew}}$. Therefore, W_{skew} is a subspace of $\mathbb{Q}^{n\times n}$.

(b) If $A \in W_{\text{skew}} \cap W_{\text{sym}}$, then $A^\mathsf{T} = A = -A$ which implies that $2A = 0$ and $A = 0$. Hence, we have $W_{\text{skew}} \cap W_{\text{sym}} = \{0\}$ and the subspaces W_{skew}, W_{sym} are independent. Given $B \in \mathbb{Q}^{n\times n}$, using $(B^\mathsf{T})^\mathsf{T} = B$, we obtain the following equations:

$$\left(\tfrac{1}{2}(B - B^\mathsf{T})\right)^\mathsf{T} = \tfrac{1}{2}(B^\mathsf{T} - B) = -\tfrac{1}{2}(B - B^\mathsf{T})\,, \tag{$\dagger$}$$

$$\left(\tfrac{1}{2}(B + B^\mathsf{T})\right)^\mathsf{T} = \tfrac{1}{2}(B^\mathsf{T} + B) = \tfrac{1}{2}(B + B^\mathsf{T})\,, \tag{$\ddagger$}$$

$$B = \tfrac{1}{2}(B - B^\mathsf{T}) + \tfrac{1}{2}(B + B^\mathsf{T})\,. \tag{$\star$}$$

The equation ($\dagger$) shows $\frac{1}{2}(B - B^\mathsf{T}) \in W_{\text{skew}}$, ($\ddagger$) shows $\frac{1}{2}(B + B^\mathsf{T}) \in W_{\text{sym}}$ and ($\star$) establishes that $\mathbb{Q}^{n\times n} = W_{\text{skew}} + W_{\text{sym}}$. Since W_{skew}, W_{sym} are independent and $\mathbb{Q}^{n\times n} = W_{\text{skew}} + W_{\text{sym}}$, we conclude that $\mathbb{Q}^{n\times n} = W_{\text{skew}} \oplus W_{\text{sym}}$. □

2. Prove or give a counterexample to following: if U_1 and U_2 are finite dimensional subspaces of V, then $U_1 + U_2$ is finite dimensional, and $\dim(U_1 + U_2) \leq \dim U_1 + \dim U_2$.

Solution. We prove the statement. By definition of finite dimensional, there exist finite lists of vectors $(u_1, \dots, u_n)$ and $(v_1, \dots, v_m)$ in V such that $U_1 = \operatorname{span}(u_1, \dots, u_n)$ and $U_2 = \operatorname{span}(v_1, \dots, v_m)$; in fact, we can assume these have already been reduced to bases. It is straightforward to show that $U_1 + U_2 = \operatorname{span} \mathcal{B}$, where $\mathcal{B} = (u_1, \dots, u_n, v_1, \dots, v_m)$. Thus, $\mathcal{B}$ is a spanning list for $U_1 + U_2$. Because this list can be reduced to a basis, we find that $\dim(U_1 + U_2) \leq n + m = \dim U_1 + \dim U_2$. □

3. Let $P := \mathbb{R}[t]_{\leq 2}$ be the real vector space of all polynomial functions of degree at most 2 and consider $V := \mathbb{R}^P$, the real vector space of all functions from P to $\mathbb{R}$. Determine the linear independence or dependence of the following lists (f_1, f_2, f_3) in V.

(a) for $p \in P$, let $f_1(p) := p(0)$, $f_2(p) := p(1)$ and $f_3(p) := p(2)$;
(b) for $p \in P$, let $f_1(p) := p(0)$, $f_2(p) := \int_0^1 p(t)\, dt$ and $f_3(p) := \int_{-1}^1 p(t)\, dt$.

Solution. To show that (f_1, f_2, f_3) is linearly independent, it suffices to find polynomials $p_1, p_2, p_3 \in P$ such that $f_i(p_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ for all $1 \leq i \leq 3$ and all $1 \leq j \leq 3$. Indeed, given such polynomials, a linear relation $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$ with $a_1, a_2, a_3 \in \mathbb{R}$ yields

$$\begin{aligned} 0 &= a_1 f_1(p_1) + a_2 f_2(p_1) + a_3 f_3(p_1) = a_1 \\ 0 &= a_1 f_1(p_2) + a_2 f_2(p_2) + a_3 f_3(p_2) = a_2 \\ 0 &= a_1 f_1(p_3) + a_2 f_2(p_3) + a_3 f_3(p_3) = a_3 \end{aligned}$$

which means $a_1 = a_2 = a_3 = 0$ and (f_1, f_2, f_3) is linearly independent.

(a) If $p_1(t) = \frac{1}{2}(t-1)(t-2)$, $p_2(t) = t(2-t)$, and $p_3(t) = \frac{1}{2}t(t-1)$, then we have

$$\begin{aligned} f_1(p_1) &= p_1(0) = \tfrac{1}{2}(0-1)(0-2) = 1 & f_1(p_2) &= p_2(0) = \tfrac{1}{2}(0)(2-0) = 0 \\ f_1(p_3) &= p_3(0) = \tfrac{1}{2}(0)(0-1) = 0 & f_2(p_1) &= p_1(1) = \tfrac{1}{2}(1-1)(1-2) = 0 \\ f_2(p_2) &= p_2(1) = (1)(2-1) = 1 & f_2(p_3) &= p_3(1) = \tfrac{1}{2}(1)(1-1) = 0 \\ f_3(p_1) &= p_1(2) = \tfrac{1}{2}(2-1)(2-2) = 0 & f_3(p_2) &= p_2(2) = 2(2-2) = 0 \\ f_3(p_3) &= p_3(2) = \tfrac{1}{2}(2)(2-1) = 1\,. \end{aligned}$$

(b) If $p_1(t) = -3t^2 + 1$, $p_2(t) = 2t$, and $p_3(t) = \frac{3}{2}t^2 - t$, then we have

$$\begin{aligned} f_1(p_1) &= p_1(0) = 1 & f_1(p_2) &= p_2(0) = 0 \\ f_1(p_3) &= p_3(0) = 0 & f_2(p_1) &= \textstyle\int_0^1 -3t^2 + 1\; dt = [-t^3 + t]_0^1 = 0 \\ f_2(p_2) &= \textstyle\int_0^1 2t\; dt = [t^2]_0^1 = 1 & f_2(p_3) &= \textstyle\int_0^1 \frac{3}{2}t^2 - t\; dt = [\frac{1}{2}t^3 - \frac{1}{2}t^2]_0^1 = 0 \\ f_3(p_1) &= \textstyle\int_{-1}^1 -3t^2 + 1\; dt = [-t^3 + t]_{-1}^1 = 0 & f_3(p_2) &= \textstyle\int_{-1}^1 2t\; dt = [t^2]_{-1}^1 = 0 \\ f_3(p_3) &= \textstyle\int_{-1}^1 \frac{3}{2}t^2 - t\; dt = [\frac{1}{2}t^3 - \frac{1}{2}t^2]_{-1}^1 = 1\,. \end{aligned}$$

Therefore, both lists (f_1, f_2, f_3) in V are linearly independent. $\square$