Solutions #3

1. The *conjugate transpose* of a complex $(m \times n)$-matrix Z is the $(n \times m)$-matrix Z^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. A complex $(n \times n)$-matrix Z is *Hermitian* if $Z = Z^*$.

(a) Show that the Hermitian matrices form a real vector space.
(b) Find a basis for this space and determine its dimension.

Solution.

(a) Since the set of complex $(n \times n)$-matrices forms a real vector space, it suffices to show that the set of Hermitian matrices forms a real subspace. Since the zero matrix is equal to its conjugate transpose, the set of Hermitian matrices is nonempty. If X, Y are Hermitian matrices and $c \in \mathbb{R}$ (in particular $\overline{c} = c$), then we have $(cX + Y)^* = \overline{(cX+Y)^t} = \overline{cX^t + Y^t} = cX^* + Y^* = cX + Y$ and $cX + Y$ is also a Hermitian matrix. Therefore, the set of Hermitian matrices forms a real vector subspace.

(b) If $Z \in \mathbb{C}^{n \times n}$, then there exists unique real matrices A, B such that $Z = A + iB$. If Z is Hermitian, then

$$A + iB = Z = Z^* = (A+iB)^* = \overline{(A+iB)^t} = \overline{A^t + iB^t} = A^t - iB^t \,.$$

Taking the real and imaginary parts, we have $A = A^t$ and $B^t = -B^t$. Let $E_{j,k}$ denote the matrix whose only nonzero entry is a 1 in the jth row and kth column. If $A = [a_{j,k}]$ and $B = [b_{j,k}]$ then we have

$$A = \left(\sum_{j=1}^{n} a_{j,j} E_{j,j} \right) + \left(\sum_{j=1}^{n} \sum_{k=j+1}^{n} a_{j,k}(E_{j,k} + E_{k,j}) \right) = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{1,2} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{bmatrix}$$

$$B = \sum_{j=1}^{n} \sum_{k=j+1}^{n} b_{j,k}(E_{j,k} - E_{k,j}) = \begin{bmatrix} 0 & b_{1,2} & \cdots & b_{1,n} \\ -b_{1,2} & 0 & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1,n} & -b_{2,n} & \cdots & 0 \end{bmatrix} .$$

It follows that the set

$$\{E_{j,j} : 1 \leq j \leq n\} \cup \{E_{j,k} + E_{k,j} : 1 \leq j < k \leq n\} \cup \{i(E_{j,k} - E_{k,j}) : 1 \leq j < k \leq n\}$$

spans the real vector space of Hermitian matrices. If

$$0 = A + iB = \left(\sum_{j=1}^{n} a_{j,j} E_{j,j} \right) + \left(\sum_{j=1}^{n} \sum_{k=i+1}^{n} a_{j,k}(E_{j,k} + E_{k,j}) \right) + i \left(\sum_{j=1}^{n} \sum_{k=i+1}^{n} b_{j,k}(E_{j,k} - E_{k,j}) \right)$$
$$= \begin{bmatrix} a_{1,1} & a_{1,2} + ib_{1,2} & \cdots & a_{1,2} + ib_{1,n} \\ a_{1,2} - ib_{1,2} & a_{2,2} & \cdots & a_{2,n} + ib_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} - ib_{1,n} & a_{2,n} - ib_{2,n} & \cdots & a_{n,n} \end{bmatrix},$$

then $a_{j,k} = 0$ for $1 \leq j \leq k \leq n$ and $b_{j,k} = 0$ for $1 \leq j < k \leq n$, and we deduce that this set is linearly independent. Therefore, the set

$$\{E_{j,j} : 1 \leq j \leq n\} \cup \{E_{j,k} + E_{k,j} : 1 \leq j < k \leq n\} \cup \{i(E_{j,k} - E_{k,j}) : 1 \leq j < k \leq n\}$$

forms a basis for the real vector space of Hermitian matrices. Since we have

$$\begin{aligned} \left|\{E_{j,j} : 1 \leq j \leq n\}\right| &= n \\ \left|\{E_{j,k} + E_{k,j} : 1 \leq j < k \leq n\}\right| &= (n-1) + (n-2) + \cdots + 2 + 1 = \tfrac{n(n-1)}{2} \\ \left|\{i(E_{j,k} - E_{k,j}) : 1 \leq j < k \leq n\}\right| &= (n-1) + (n-2) + \cdots + 2 + 1 = \tfrac{n(n-1)}{2}, \end{aligned}$$

the vector space of Hermitian matrices has dimension $n + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n^2$. □

2(a). Let m be a positive integer and suppose $q_0, q_1, \ldots, q_m$ are polynomials in $\mathbb{F}[t]_{\leq m}$ such that $q_j(-3) = 0$ for $0 \leq j \leq m$. Show that $(q_0, q_1, \ldots, q_m)$ is linear dependent.

Solution. Suppose $(q_0, \ldots, q_m)$ is linearly independent. Since $\dim \mathbb{F}[t]_{\leq m} = m+1$, it follows that $(q_0, \ldots, q_m)$ is a basis for $\mathbb{F}[t]_{\leq m}$. Hence, $t \in \operatorname{span}(q_0, \ldots, q_m)$, which means that there exist $b_0, \ldots, b_m \in \mathbb{F}$ such that $t = b_0 q_0(t) + \cdots + b_m q_m(t)$. However, evaluating both sides of this equation at $t = -3$ yields $-3 = b_0 q_0(-3) + \cdots + b_m q(-3) = 0$, which is absurd. Therefore, $(q_0, q_1, \ldots, q_m)$ is linearly dependent. □

2(b). Prove or disprove: there exists a basis (p_0, p_1, p_2, p_3) of $\mathbb{F}[t]_{\leq 3}$ such that all of the polynomials p_0, p_1, p_2, p_3 have degree 3.

Solution. There exists a basis of $\mathbb{F}[t]_{\leq 3}$ such that all of the vectors has degree 3. For example, we claim that $p_0 = t^3 + 1$, $p_1 = t^3 + t$, $p_2 = t^3 + t^2$, $p_3 = t^3$ form a basis for $\mathbb{F}[t]_{\leq 3}$. Since $\dim \mathbb{F}[t]_{\leq 3} = 4$, it suffices to show that (p_0, p_1, p_2, p_3) is linearly independent.

If $a_0, \ldots, a_3 \in \mathbb{F}$ satisfy

$$\begin{aligned} 0 = a_0p_0 + a_1p_1 + a_2p_2 + a_3p_3 &= a_0(t^3+1) + a_1(t^3+t) + a_2(t^3+t^2) + a_3(t^3) \\ &= a_0 + a_1t + a_2t^2 + (a_0+a_1+a_2+a_3)t^3 \end{aligned}$$

then we have

$$\left\{\begin{matrix} a_0 & & & = 0 \\ & a_1 & & = 0 \\ & & a_2 & = 0 \\ a_0 + a_1 + a_2 + a_3 & & & = 0 \end{matrix}\right\} \qquad \Longrightarrow \qquad a_0 = a_1 = a_2 = a_3 = 0\,.$$

Hence, (p_0, p_1, p_2, p_3) is linearly independent. □

3. Suppose $T \in \operatorname{Hom}(V, \mathbb{F})$ and the vector $u \in V$ does not lie in $\operatorname{Ker}(T)$. Prove that $V = \operatorname{Ker}(T) \oplus \operatorname{span}(u)$.

Solution. To establish that $V = \operatorname{Ker}(T) \oplus \operatorname{span}(u)$, we show that $\operatorname{Ker}(T) \cap \operatorname{span}(u) = \{0\}$ and $V = \operatorname{Ker}(T) + \operatorname{span}(u)$.

Suppose $v \in \operatorname{Ker}(T) \cap \operatorname{span}(u)$. Since $v \in \operatorname{span}(u)$, there exists $a \in \mathbb{F}$ such that $v = au$. As $v \in \operatorname{Ker}(T)$, we have $0 = T(v) = T(au) = aT(u)$. Because $T(u) \neq 0$, it follows that $a = 0$. Thus, $\operatorname{Ker}(T) \cap \operatorname{span}(u) = \{0\}$.

Suppose $v \in V$. Because $T(u) \neq 0$, we have

$$v = \left(v - \frac{T(v)}{T(u)}u\right) + \frac{T(v)}{T(u)}u \tag{$\dagger$}$$

Since $T\big(v - \frac{T(v)}{T(u)}u\big) = T(v) - \frac{T(v)}{T(u)}T(u) = 0$, it follows that $v - \frac{T(u)}{T(u)}u \in \operatorname{Ker}(T)$. We clearly have $\frac{T(v)}{T(u)}u \in \operatorname{span}(u)$. Hence equation $(\dagger)$ proves that $V = \operatorname{Ker}(T) + \operatorname{span}(u)$. □