Solutions #4

1. Let $L\colon C^2([0,1]) \to C([0,1])$ be defined by $Lf = f''$.

(a) Show that L has no left inverses.

(b) Show that the operators G_1 and G_2, defined as follows, are right inverses:

$$(G_1 f)(x) = \int_0^x (x-t)f(t)\,dt\,,$$

$$(G_2 f)(x) = \int_0^1 g(x,y)f(y)\,dy\,, \qquad \text{where } g(x,y) = \begin{cases} x(y-1) & x < y \\ y(x-1) & y \le x\,. \end{cases}$$

(c) Let U_1 be the set of functions in $C^2([0,1])$ satisfying $f(0) = f'(0) = 0$. Show that $G_1 = L^{-1}$ if the domain of L is restricted to U_1.

(d) Let U_2 be the set of functions in $C^2([0,1])$ satisfying $f(0) = f(1) = 0$. Show that $G_2 = L^{-1}$ if the domain of L is restricted to U_2.

Solution.

(a) Suppose L has a left inverse. By definition, this means that there exists $H\colon C([0,1]) \to C^2([0,1])$ such that $HL = I$. In particular, if $Lf = Lg$ then applying H to this equation yields $f = If = HLf = HLg = Ig = g$ which implies that L is injective. However, $L(f) = f'' = L(f+1)$ shows that L is not injective. Therefore, L has no left inverse.

(b) Since

$$\begin{aligned}(LG_1 f)(x) &= \frac{d^2}{dx^2}\int_0^x (x-t)f(t)\,dt = \frac{d}{dx}\left(\frac{d}{dx}\int_0^x (x-t)f(t)\,dt\right) \\ &= \frac{d}{dx}\left((x-x)f(x) + \int_0^x \left[\frac{d}{dx}(x-t)\right] f(t)\,dt\right) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)\,,\end{aligned}$$

we see that $LG_1 = I$ and G_1 is a right inverse of L. Similarly, because

$$\begin{aligned}(LG_2 f)(x) &= \frac{d^2}{dx^2}\int_0^1 g(x,y)f(y)\,dy = \frac{d^2}{dx^2}\left(\int_0^x y(x-1)f(y)\,dy + \int_x^1 x(y-1)f(y)\,dy\right) \\ &= \frac{d^2}{dx^2}\left((x-1)\int_0^x yf(y)\,dy + x\int_x^1 (y-1)f(y)\,dy\right) \\ &= \frac{d}{dx}\left(\int_0^x yf(y)\,dy + (x-1)xf(x) + \int_x^1 (y-1)f(y)\,dy - x(x-1)f(x)\right) \\ &= \frac{d}{dx}\left(\int_0^1 yf(y)\,dy + \int_1^x f(y)\,dy\right) = f(x)\,,\end{aligned}$$

it follows that $LG_2 = I$ and G_2 is a right inverse of L.

(c) If $f \in U_1$, then we have

$$(G_1 L f)(x) = \int_0^x (x-t) f''(t)\, dt = \big[(x-t) f'(t)\big]_0^x - \int_0^x f'(t)(-1)\, dt$$
$$= (x-x)f'(x) + (x-0)f'(0) + \big[f(t)\big]_0^x = f(x) - f(0) = f(x)\,,$$

so $G_1 L = I$. Combining this with part (b), we see that $G_1 = L^{-1}$ if the domain of L is restricted to U_1.

(d) If $f \in U_2$, then we have

$$(G_2 L f)(x) = \int_0^1 g(x,y) f''(y)\, dy = \int_0^x y(x-1) f''(y)\, dy + \int_x^1 x(y-1) f''(y)\, dy$$
$$= (x-1)\int_0^x y f''(y)\, dy + x \int_x^1 (y-1) f''(y)\, dy$$
$$= (x-1)\left\{ \big[y f'(y)\big]_0^x - \int_0^x f'(y)\, dy \right\} + x \left\{ \big[(y-1) f'(y)\big]_x^1 - \int_x^1 f'(y)\, dy \right\}$$
$$= (x-1)x f'(x) - (x-1)\big[f(y)\big]_0^x - x(x-1) f'(x) - x\big[f(y)\big]_x^1$$
$$= -(x-1) f(x) + (x-1) f(0) - x f(1) + x f(x) = f(x)\,,$$

so $G_2 L = I$. Combining this with part (b), we see that $G_2 = L^{-1}$ if the domain of L is restricted to U_2. □

2. Let V be a finite dimensional vector space and consider $S, T \in \operatorname{End}(V)$.

(a) Show that ST is invertible if and only if both S and T are invertible.

(b) Prove that $ST = I$ if and only if $TS = I$.

(c) Give an example illustrating that both (a) and (b) are false over an infinite dimensional vector space.

Solution.

(a) $\Longrightarrow$ Suppose ST is invertible. Hence, ST is bijective which means $\operatorname{Ker} ST = \{0\}$ and $\operatorname{Im} ST = V$. Since $\operatorname{Im} ST \subseteq \operatorname{Im} S$ and $\operatorname{Ker} T \subseteq \operatorname{Ker} ST$, we deduce that $\operatorname{Im} S = V$ and $\operatorname{Ker} T = \{0\}$. For a linear operator on a finite-dimensional vector space, being invertible is equivalent to being injective or being surjective. Therefore, S and T are invertible.

$\Longleftarrow$ If S and T are invertible, then we have $(T^{-1}S^{-1})(ST) = T^{-1}T = I$ and $(ST)(T^{-1}S^{-1}) = SS^{-1} = I$. Therefore, the inverse of ST is $T^{-1}S^{-1}$.

(b) By symmetry, it suffices to prove that $ST = I$ implies that $TS = I$. If $ST = I$, then $\operatorname{Ker} T \subseteq \operatorname{Ker} ST = \{0\}$ and T is injective. For a linear operator on a finite-dimensional vector space, being invertible is equivalent to being injective. Hence,

T is invertible and we have

$$TS = TS(I) = TS(TT^{-1}) = T(ST)T^{-1} = TIT^{-1} = TT^{-1} = I\,.$$

(c) Suppose $V = \mathbb{R}[t]$ and let $S, T \in \operatorname{End}(V)$ be defined by $(Sf)(t) = f'(t)$ and $(Tf)(t) = \int_0^t f(y)\ dy$ for $f \in V$. The fundamental theorem of calculus shows that

$$(STf)(t) = \frac{d}{dt}\int_0^t f(y)\ dy = f(t) \qquad \text{and} \qquad (TSf)(t) = \int_0^t f'(y)\ dy = f(t) - f(0)\,,$$

which implies $ST = I$ and $TS \neq I$. In particular, over an infinite dimensional vector space, both parts (a) and (b) are false. □

3. Define $J\colon \mathbb{R}[t]_{\leq 2} \to \mathbb{R}[t]_{\leq 2}$ by $(Jp)(t) = \dfrac{1}{2}\displaystyle\int_{-1}^{1} (6 + 9st - 15s^2t^2)p(s)\ ds$.

(a) Find the matrix $\mathcal{M}(J)$ with respect to the basis $(1, t, t^2)$.
(b) Find a basis for $\operatorname{Ker} J$ and $\operatorname{Im} J$.
(c) Show that J^{-1} exists and find an expression for $J^{-1}(a + bt + ct^2)$.
(d) Find p such that $J(p) = (1 + t)^2$.
(e) Find q such that $J^2(q) = t^2$.

Solution.

(a) Since we have

$$(J1)(t) = \frac{1}{2}\int_{-1}^{1} (6 + 9st - 15s^2t^2)\ ds = \tfrac{1}{2}\big[6s + (9/2)s^2t - 5s^3t^2\big]_{-1}^{1} = 6 - 5t^2$$

$$(Jt)(t) = \frac{1}{2}\int_{-1}^{1} (6 + 9st - 15s^2t^2)s\ ds = \tfrac{1}{2}\big[3s^2 + 3s^3t - \tfrac{15}{4}s^4t^2\big]_{-1}^{1} = 3t$$

$$(Jt^2)(t) = \frac{1}{2}\int_{-1}^{1} (6 + 9st - 15s^2t^2)s^2\ ds = \tfrac{1}{2}\big[2s^3 + (9/4)s^4t - 3s^5t^2\big]_{-1}^{1} = 2 - 3t^2\,,$$

it follows that

$$\mathcal{M}(J) = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 3 & 0 \\ -5 & 0 & -3 \end{bmatrix}.$$

(b) Row-reducing $\mathcal{M}(J)$ yields

$$\begin{bmatrix} 6 & 0 & 2 \\ 0 & 3 & 0 \\ -5 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{4}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\operatorname{Ker}\mathcal{M}(J) = \{0\}$, so $\operatorname{Ker} J = \{0\}$ and J is injective. Because an injective endomorphism on a finite-dimensional vector space is invertible, we see that $\operatorname{Im} J = \mathbb{R}[t]_{\leq 2}$.

(c) Part (b) shows J is invertible. To compute J^{-1}, we first compute $\mathcal{M}(J)^{-1}$. The matrix of minors of $\begin{bmatrix} 6 & 0 & 2 \\ 0 & 3 & 0 \\ -5 & 0 & -3 \end{bmatrix}$ equals $\begin{bmatrix} -9 & 0 & 15 \\ 0 & -8 & 0 \\ -6 & 0 & 18 \end{bmatrix}$, which also happens to be the cofactor matrix. The adjugate matrix is the transpose $\begin{bmatrix} -9 & 0 & -6 \\ 0 & -8 & 0 \\ 15 & 0 & 18 \end{bmatrix}$. Now we have

$$\begin{bmatrix} -9 & 0 & -6 \\ 0 & -8 & 0 \\ 15 & 0 & 18 \end{bmatrix} \begin{bmatrix} 6 & 0 & 2 \\ 0 & 3 & 0 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix},$$

so the inverse is $\mathcal{M}(J)^{-1} = (1/24)\begin{bmatrix} 9 & 0 & 6 \\ 0 & 8 & 0 \\ -15 & 0 & -18 \end{bmatrix}$

Since

$$\mathcal{M}(J)^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 9 & 0 & 6 \\ 0 & 8 & 0 \\ -15 & 0 & -18 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 9a+6c \\ 8b \\ -15a-18c \end{bmatrix},$$

it follows that $J^{-1}(a + bt + ct^2) = \frac{1}{24}\big((9a + 6c) + (8b)t + (-15a - 18c)t^2\big)$.

(d) Part (c) yields $J^{-1}\big((1+t)^2\big) = J^{-1}(1 + 2t + t^2) = \frac{1}{24}(15 + 8t - 33t^2)$ so $J(\frac{1}{24}(15 + 8t - 33t^2)) = (1 + t)^2$.

(e) Using part (c) twice, we obtain $J^{-1}(t^2) = 4 - 12t^2$ and $J^{-1}\big(4 - 12t^2\big) = -\frac{3}{2} + \frac{13}{2}t^2$. Hence, $J^2\big(-\frac{3}{2} + \frac{13}{2}t^2\big) = t^2$. □