Solutions #5

1. Suppose that $a_0, \dots, a_m$ are distinct elements in $\mathbb{F}$ and that $b_0, \dots, b_m$ are elements in $\mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathbb{F}[t]_{\leq m}$ such that $p(a_j) = b_j$ for $0 \leq j \leq m$.

Solution. Consider $T \in \operatorname{Hom}(\mathbb{F}[t]_{\leq m}, \mathbb{F}^{m+1})$ defined by $T\big(p(t)\big) = \big(p(a_0), \dots, p(a_m)\big)$. If T is injective then at most one polynomial p satisfies the required condition. Moreover, if T is surjective, then at least one polynomial p satisfies the required condition. Thus, it suffices to show that T is bijective.

Suppose $q \in \operatorname{Ker} T$. Since $q(a_0) = q(a_1) = \cdots = q(a_m) = 0$, q is a polynomial of degree at most m with at least $m+1$ distinct roots which implies that $q = 0$. Hence, $\operatorname{Ker} T = \{0\}$ and T is injective.

Since $\operatorname{Im} T \subseteq \mathbb{F}^{m+1}$, the dimension formula gives

$$m + 1 = \dim \mathbb{F}[t]_{\leq m} = \dim \operatorname{Ker} T + \dim \operatorname{Im} T = 0 + \dim \operatorname{Im} T \leq \dim \mathbb{F}^{m+1} = m + 1\,.$$

Hence, $\operatorname{Im} T = \mathbb{F}^{m+1}$ and T is surjective. □

Remark. Surjectivity of T can also be established by giving an explicit construction. For example, T sends the *Lagrange polynomials*

$$f_j(t) = \frac{(t - a_0) \cdots (t - a_{j-1})(t - a_{j+1}) \cdots (t - a_m)}{(a_j - a_0) \cdots (a_j - a_{j-1})(a_j - a_{j+1}) \cdots (a_j - a_m)} = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{t - a_k}{a_j - a_k} \in \mathbb{F}[t]_{\leq m}$$

to the standard basis of $\mathbb{F}^{m+1}$.

Alternative Solution. As in the first solution, it suffices to show that T is invertible. With respect to the monomial basis on $\mathbb{F}[t]_{\leq m}$ and the standard basis on $\mathbb{F}^{m+1}$, the matrix associated to T is

$$\mathcal{M}(T) = \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^m \\ 1 & a_1 & a_1^2 & \cdots & a_1^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^m \end{bmatrix} ;$$

this is called the *Vandermonde* matrix. To show that T is invertible, it suffices to prove that $\det \mathcal{M}(T) \neq 0$. To accomplish this, we claim that

$$V(a_0, \dots, a_m) = \det \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^m \\ 1 & a_1 & a_1^2 & \cdots & a_1^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^m \end{bmatrix} = \prod_{i<j} (a_j - a_i)\,.$$

We proceed by induction on m. The case $m = 0$ is trivial. Assume $m > 0$. For each $k \geq 2$, we subtract from the k^{th} column the $(k-1)^{\text{th}}$ column multiplied by a_0; the value of the determinant is unaltered so

$$V(a_0, \dots, a_m) = \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & a_1 - a_0 & a_1(a_1 - a_0) & \cdots & a_1^{m-1}(a_1 - a_0) \\ 1 & a_2 - a_0 & a_2(a_2 - a_0) & \cdots & a_2^{m-1}(a_2 - a_0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m - a_0 & a_m(a_m - a_0) & \cdots & a_m^{m-1}(a_m - a_0) \end{bmatrix}.$$

Expanding along the first row and taking out the factor of $(a_k - a_0)$ from the kth row, we have $V(a_0, \dots, a_m) = \left(\prod_{j=1}^{m} (a_j - a_0) \right) V(a_1, \dots, a_m)$ and the induction hypothesis establishes the claim. Since $a_0, \dots, a_m$ are distinct elements in $\mathbb{F}$, we conclude that $\det \mathcal{M}(T) \neq 0$ and T is invertible. □

2. Consider $T \in \operatorname{Hom}(\mathbb{R}[x]_{\leq 2}, \mathbb{R}[x]_{\leq 2})$ defined by $(Tf)(x) = \int_{-1}^{1} (x-y)^2 f(y)\, dy - 2f(0)x^2$ for all $f \in \mathbb{R}[x]_{\leq 2}$. Find all eigenvalues and eigenvectors for T.

Solution. Fix $(1, x, x^2)$ as a basis for $\mathbb{R}[x]_{\leq 2}$. Since

$$\begin{aligned} (T1)(x) &= \int_{-1}^{1} (x-y)^2(1)\, dy - 2(1)x^2 = \int_{-1}^{1} x^2 - 2xy + y^2\, dy - 2x^2 \\ &= \left[x^2 y - xy^2 + \tfrac{1}{3}y^3\right]_{-1}^{1} - 2x^2 = \tfrac{2}{3} \\ (Tx)(x) &= \int_{-1}^{1} (x-y)^2(y)\, dy - 2(0)x^2 = \int_{-1}^{1} x^2 y - 2xy^2 + y^3\, dy \\ &= \left[\tfrac{1}{2}x^2y^2 - \tfrac{2}{3}xy^3 + \tfrac{1}{4}y^4\right]_{-1}^{1} = -\tfrac{4}{3}x \\ (Tx^2)(x) &= \int_{-1}^{1} (x-y)^2(y^2)\, dy - 2(0)x^2 = \int_{-1}^{1} x^2y^2 - 2xy^3 + y^4\, dy \\ &= \left[\tfrac{1}{3}x^2y^3 - \tfrac{1}{2}xy^4 + \tfrac{1}{5}y^5\right]_{-1}^{1} = \tfrac{2}{5} + \tfrac{2}{3}x^2, \end{aligned}$$

we have $\mathcal{M}(T) = \left[\begin{smallmatrix} 2/3 & 0 & 2/5 \\ 0 & -4/3 & 0 \\ 0 & 0 & 2/3 \end{smallmatrix}\right]$. Since this matrix is upper-triangular, it follows that the eigenvalues for T are the diagonal entries, namely $2/3$ and $-4/3$. Moreover, row-reduction yields

$$\mathcal{M}\left(\tfrac{2}{3}I - T\right) = \left[\begin{smallmatrix} 0 & 0 & -2/5 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] \qquad \mathcal{M}\left(-\tfrac{4}{3}I - T\right) = \left[\begin{smallmatrix} -2 & 0 & -2/5 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$$

we see that the nonzero vectors in $\operatorname{Ker}\left(\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = \operatorname{span}\left(\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]\right)$ are the eigenvectors of $\mathcal{M}(T)$ corresponding to $2/3$ and the nonzero vectors in $\operatorname{Ker}\left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = \operatorname{span}\left(\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]\right)$ are the eigenvectors of $\mathcal{M}(T)$ corresponding to $-4/3$. Therefore, the nonzero vectors in $\operatorname{span}(1)$ are the eigenvectors of T corresponding to $2/3$ and the nonzero vectors in $\operatorname{span}(x)$ are the eigenvectors of T corresponding to $-4/3$. $\square$

3. Suppose $T \in \operatorname{End}(\mathbb{F}^n)$ satisfies $T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$. Find all eigenvalues and eigenvectors of T.

Solution. Let $(e_1, \dots, e_n)$ be the standard basis for $\mathbb{F}^n$; specifically, e_i is the column vector with 1 in the ith spot and zero elsewhere. Set $v_1 = e_1 + \cdots + e_n$ and, for $2 \leq k \leq n$, set $v_k := e_1 - e_k$. Since $Tv_1 = (1 + \cdots + 1, \dots, 1 + \cdots + 1) = (n, \cdots, n) = nv_1$, we see that v_1 is an eigenvector corresponding to the eigenvalue n. Similarly, for $2 \leq k \leq n$, we have $Tv_k = (1-1, \dots, 1-1) = (0, \dots, 0) = 0v_k$ which implies that v_k is an eigenvector corresponding to the eigenvalue 0. Given a linear relation

$$0 = a_1 v_1 + \cdots + a_n v_n = (a_1 + a_2 + a_3 + \cdots + a_n)e_1 + (a_1 - a_2)e_2 + \cdots + (a_1 - a_n)e_n \,,$$

we obtain the system of linear equations

$$\begin{array}{ccccccccccc} a_1 & + & a_2 & + & a_3 & + & \cdots & + & a_n & = & 0 \\ a_1 & - & a_2 & & & & & & & = & 0 \\ a_1 & & & - & a_3 & & & & & = & 0 \\ \vdots & & & & & & \ddots & & & & \vdots \\ a_1 & & & & & & & - & a_n & = & 0 \end{array}$$

which implies $a_1 = a_2 = \cdots = a_n = 0$. Since this relation is trival, $(v_1, \dots, v_n)$ is linearly independent and forms a basis of $\mathbb{F}^n$. Therefore, the eigenvalues of T are n and 0 (which has multiplicity $n-1$). The nonzero vectors in $\operatorname{span}(v_1)$ are eigenvectors corresponding to n and the nonzero vectors in $\operatorname{span}(v_2, \dots, v_n)$ are the eigenvectors corresponding to 0. Moreover, the matrix of T with respect to $(v_1, \dots, v_n)$ is

$$\mathcal{M}(T) = \begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} .$$

$\square$