# Solutions #6

**1.** Let $V$ be a complex inner product space. For $v, w \in V$ prove that $\langle v, w\rangle = 0$ if and only if $\|v\| \le \|v + aw\|$ for all $a \in \mathbb{C}$.

*Solution.*

$\Longrightarrow$: Suppose $\langle v, w\rangle = 0$. Since $|a|\|w\|^2 \ge 0$ for all $a \in \mathbb{C}$, we have

$$\|v+aw\|^2 = \langle v+aw, v+aw\rangle = \langle v, v\rangle + \overline{a}\langle v, w\rangle + a\langle w, v\rangle + a\overline{a}\langle w, w\rangle = \|v\|^2 + |a|\|w\|^2 \ge \|v\|^2 .$$

Taking square roots yields $\|v\| \le \|v + aw\|$ for all $a \in \mathbb{C}$.

$\Longleftarrow$: If $w = 0$ then $\langle v, w\rangle = 0$, so we may assume $w \ne 0$. We can express $v$ as a scalar multiple of $w$ plus a vector orthogonal to $w$ as follows: $v = \frac{\langle v,w\rangle}{\|w\|^2} w + \big(v - \frac{\langle v,w\rangle}{\|w\|^2} w\big)$. Taking the magnitude and applying the Pythagorean Theorem yields

$$\|v\|^2 = \left\| \tfrac{\langle v,w\rangle}{\|w\|^2} w + \big(v - \tfrac{\langle v,w\rangle}{\|w\|^2} w\big) \right\|^2 = |\langle v, w\rangle|^2 + \left\| v + \big(-\tfrac{\langle v,w\rangle}{\|w\|^2}\big) w \right\|^2 .$$

Since $\|v\| \le \|v + aw\|$ for all $a \in \mathbb{C}$, we obtain $\left\| v + \big(-\frac{\langle v,w\rangle}{\|w\|^2}\big) w \right\| \ge \|v\|$ and

$$\|v\|^2 = |\langle v, w\rangle|^2 + \left\| v + \big(-\tfrac{\langle v,w\rangle}{\|w\|^2}\big) w \right\|^2 \ge |\langle v, w\rangle|^2 + \|v\|^2 .$$

Therefore $0 \ge |\langle v, w\rangle|^2$ which implies $\langle v, w\rangle = 0$. □

**2.** Prove the *polar identities*.

**(a)** On a real inner product space $V$, show that for all $v, w \in V$, we have

$$\langle v, w\rangle = \tfrac{1}{4}\big(\|v + w\|^2 - \|v - w\|^2\big) .$$

**(b)** On a complex inner product space $V$, show that for all $v, w \in V$, we have

$$\langle v, w\rangle = \tfrac{1}{4}\big[\|v + w\|^2 - \|v - w\|^2 + i\big(\|v + iw\|^2 - \|v - iw\|^2\big)\big] .$$

*Solution.*

**(a)** We have

$$\begin{aligned}
\tfrac{1}{4}\big(\|v + w\|^2 - \|v - w\|^2\big) &= \tfrac{1}{4}\big(\langle v + w, v + w\rangle - \langle v - w, v - w\rangle\big) \\
&= \tfrac{1}{4}\big(\langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle - \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle - \langle w, w\rangle\big) \\
&= \tfrac{1}{4}\big(2\langle v, w\rangle + 2\langle w, v\rangle\big) = \langle v, w\rangle .
\end{aligned}$$

**(b)** Similarly, we have

$$
\begin{aligned}
&\tfrac{1}{4}\big[\|v+w\|^2-\|v-w\|^2+i\big(\|v+iw\|^2-\|v-iw\|^2\big)\big]\\
&\quad=\tfrac{1}{4}\big[\langle v+w,v+w\rangle-\langle v-w,v-w\rangle+i\big(\langle v+iw,v+iw\rangle-\langle v-iw,v-iw\rangle\big)\big]\\
&\quad=\tfrac{1}{4}\big[\langle v,v\rangle+\langle v,w\rangle+\langle w,v\rangle+\langle w,w\rangle-\langle v,v\rangle+\langle v,w\rangle+\langle w,v\rangle-\langle w,w\rangle\\
&\qquad\quad+i\langle v,v\rangle+i\langle v,iw\rangle+i\langle iw,v\rangle+i\langle iw,iw\rangle-i\langle v,v\rangle-i\langle v,-iw\rangle-i\langle -iw,v\rangle-i\langle -iw,-iw\rangle\big]\\
&\quad=\tfrac{1}{4}\big[2\langle v,w\rangle+2\langle w,v\rangle+\langle v,w\rangle-\langle w,v\rangle+\langle v,w\rangle-\langle w,v\rangle\big]=\tfrac{1}{4}\big[4\langle v,w\rangle\big]=\langle v,w\rangle\,. \qquad\square
\end{aligned}
$$

**3.** Let $V$ be the $\mathbb{R}$-vector space of continuous functions over $[-1,1]$ with inner product $\langle f,g\rangle=\int_{-1}^{1} f(t)g(t)\;dt$. Find the quartic (i.e. degree $4$) polynomial $g(t)$ that best fits the function $f(t)=\cos(\pi t)$ over $[-1,1]$.

*Solution.* An orthonormal basis of $U=\mathbb{F}[t]_{\le 4}$ is obtained by applying the Gram–Schmidt algorithm to the basis $(1,t,t^2,t^3,t^4)$. From an example in class, the resulting basis is given by the first five Legendre polynomials $\big(f_0(t),f_1(t),f_2(t),f_3(t),f_4(t),f_5(t)\big)$. The formula $f_k(t)=\left(\frac{\sqrt{2k+1}}{\sqrt{2}}\right)\frac{1}{2^k k!}\frac{d^k}{dt^k}(t^2-1)^k$ yields $f_0(t)=\frac{1}{\sqrt{2}}$, $f_1(t)=\frac{\sqrt{3}}{\sqrt{2}}t$, $f_2(t)=\frac{\sqrt{5}}{2\sqrt{2}}(3t^2-1)$, $f_3(t)=\frac{\sqrt{7}}{2\sqrt{2}}(5t^3-3t)$, and $f_4(t)=\frac{3}{8\sqrt{2}}(35t^4-30t^2+3)$. The polynomial that best fits $f(t)=\cos(\pi t)$ is the projection $P_U(f)$, given by

$$
\begin{aligned}
P_U(f)&=\langle f,f_0\rangle f_0(t)+\langle f,f_1\rangle f_1(t)+\langle f,f_2\rangle f_2(t)+\langle f,f_3\rangle f_3(t)+\langle f,f_4\rangle f_4(t)\\
&=\int_{-1}^{1}\cos(\pi t)f_0(t)\,dt\;f_0(t)+\int_{-1}^{1}\cos(\pi t)f_1(t)\,dt\;f_1(t)+\int_{-1}^{1}\cos(\pi t)f_2(t)\,dt\;f_2(t)\\
&\quad+\int_{-1}^{1}\cos(\pi t)f_3(t)\,dt\;f_3(t)+\int_{-1}^{1}\cos(\pi t)f_4(t)\,dt\;f_4(t)\\
&=2\int_{0}^{1}\cos(\pi t)f_0(t)\,dt\;f_0(t)+2\int_{0}^{1}\cos(\pi t)f_2(t)\,dt\;f_2(t)+2\int_{0}^{1}\cos(\pi t)f_4(t)\,dt\;f_4(t),
\end{aligned}
$$

because $f_1$ and $f_3$ are odd, $f_0$, $f_2$, and $f_4$ are even, and $\cos(\pi t)$ is even. We evaluate

$$
\begin{aligned}
\int_0^1 \cos(\pi t)\;dt&=\left.\frac{\sin(\pi t)}{\pi}\right|_0^1=0\\
\int_0^1 t^2\cos(\pi t)\;dt&=\left.\frac{(\pi^2t^2-2)\sin(\pi t)+2\pi t\cos(\pi t)}{\pi^3}\right|_0^1=-2/\pi^2\\
\int_0^1 t^4\cos(\pi t)\;dt&=\left.\frac{4\pi t(\pi^2t^2-6)\cos(\pi t)+(\pi^4t^4-12\pi^2t^2+24)\sin(\pi t)}{\pi^5}\right|_0^1=-4(\pi^2-6)/\pi^4.
\end{aligned}
$$

Thus,

$$\begin{aligned}
P_U(f) &= 2\int_0^1 \cos(\pi t) f_0(t)\,dt\ f_0(t) + 2\int_0^1 \cos(\pi t) f_2(t)\,dt\ f_2(t) + 2\int_0^1 \cos(\pi t) f_4(t)\,dt\ f_4(t) \\
&= 2\cdot 0\cdot f_0(t) + 2\left(\int_0^1 \cos(\pi t)\frac{\sqrt{5}}{2\sqrt{2}}(3t^2-1)\,dt\right) f_2(t) \\
&\qquad\qquad + 2\left(\int_0^1 \cos(\pi t)\frac{3}{8\sqrt{2}}(35t^4-30t^2+3)\,dt\right) f_4(t) \\
&= \left(\frac{3\sqrt{5}}{\sqrt{2}}\int_0^1 t^2\cos(\pi t)\,dt - \frac{\sqrt{5}}{\sqrt{2}}\int_0^1 \cos(\pi t)\,dt\right) f_2(t) \\
&\qquad + \left(\frac{105}{4\sqrt{2}}\int_0^1 t^4\cos(\pi t)\,dt - \frac{90}{4\sqrt{2}}\int_0^1 t^2\cos(\pi t) + \frac{9}{4\sqrt{2}}\int_0^1 \cos(\pi t)\right) f_4(t) \\
&= \left(\frac{3\sqrt{5}}{\sqrt{2}}(-2/\pi^2) - \frac{\sqrt{5}}{\sqrt{2}}(0)\right) f_2(t) \\
&\qquad + \left(\frac{105}{4\sqrt{2}}(-4(\pi^2-6)/\pi^4) - \frac{90}{4\sqrt{2}}(-2/\pi^2) + \frac{9}{4\sqrt{2}}(0)\right) f_4(t) \\
&= \left(\frac{-3\sqrt{10}}{\pi^2}\right) f_2(t) + \left(\frac{45}{\sqrt{2}\pi^2} - \frac{105(\pi^2-6)}{\sqrt{2}\pi^4}\right) f_4(t) \\
&\approx (-0.9612171466) f_2(t) + (0.27456800889) f_4(t)
\end{aligned}$$

Hence, we can nicely approximate $\cos(\pi t)$ on the interval $[-1,1]$ using the quartic polynomial $(-0.9612171466)f_2(t) + (0.27456800889)f_4(t)$, which approximately equals $p(t) = 2.5482t^4 - 4.4639t^2 + 0.978326$.

**Remark.** It's interesting to compare this approximation with the quartic Taylor polynomial $T(t) = 1 - \frac{\pi^2 t^2}{2} + \frac{\pi^4 t^4}{24}$. For example, $\cos(\pi/6) - T(1/6) \approx -0.00003$, while $\cos(\pi/6) - p(1/6) \approx 0.00973$, which shows that the Taylor polynomial gives a better approximation at the value $t = 1/6$. However, the polynomial we have computed is a *better overall fit* on the interval $[-1,1]$. We see this through the norm, i.e. the distance squared from $\cos(\pi t)$ to the Taylor polynomial $T(t)$ is $\int_{-1}^1 (cos(\pi t) - (1 - \frac{\pi^2 t^2}{2} + \frac{\pi^4 t^4}{24}))^2 dt \approx 0.20347$, while the distance squared from $\cos(\pi t)$ to $p(t)$ is $\int_{-1}^1 (cos(\pi t) - (2.5482t^4 - 4.4639t^2 + 0.978326))^2 dt \approx 0.00067$. $\square$